

EXPLICIT ESTIMATES FOR THE ZEROS OF HECKE L -FUNCTIONS

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ABSTRACT. Let K be a number field and, for an integral ideal \mathfrak{q} of K , let χ be a character of the narrow ray class group modulo \mathfrak{q} . We establish various new and improved explicit results, with effective dependence on K , \mathfrak{q} and χ , regarding the zeros of the Hecke L -function $L(s, \chi)$, such as zero-free regions, Deuring-Heilbronn phenomenon, and zero density estimates.

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1. INTRODUCTION

Let K be a number field of degree $n_K = [K : \mathbb{Q}]$ with absolute discriminant $d_K = |\text{disc}(K/\mathbb{Q})|$ and ring of integers \mathcal{O}_K . For an integral ideal $\mathfrak{q} \subseteq \mathcal{O}_K$, the (narrow) ray class group modulo \mathfrak{q} is defined to be $\text{Cl}(\mathfrak{q}) := I(\mathfrak{q})/P_{\mathfrak{q}}$ where $I(\mathfrak{q})$ is the group of fractional ideals of K relatively prime to \mathfrak{q} , and $P_{\mathfrak{q}}$ is the subgroup of principal ideals (α) of K such that $\alpha \equiv 1 \pmod{\mathfrak{q}}$. Recall $\alpha \equiv 1 \pmod{\mathfrak{q}}$ if and only if α is totally positive and $v(\alpha - 1) \geq v(\mathfrak{q})$ for all discrete valuations v of K/\mathbb{Q} . Characters χ of the ray class group will be referred to as Hecke characters, which we will often denote $\chi \pmod{\mathfrak{q}}$. Let $\text{ord } \chi$ be the multiplicative order of χ in $\text{Cl}(\mathfrak{q})$.

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A Hecke character $\chi \pmod{\mathfrak{q}}$ possesses an associated L -function defined by

$$L(s, \chi) := \prod_{\mathfrak{p} \nmid \mathfrak{q}} \left(1 - \frac{\chi(\mathfrak{p})}{(\mathbf{N}\mathfrak{p})^s}\right)^{-1} \quad \text{for } \sigma > 1,$$

where $s = \sigma + it$, $\mathbf{N} = \mathbf{N}_{\mathbb{Q}}^K$ is the absolute norm on integral ideals of K , and the product is over prime ideals $\mathfrak{p} \subseteq \mathcal{O}_K$. In the special case $\mathfrak{q} = \mathcal{O}_K$ and $\chi = \chi_0$ the principal character, the associated L -function is the Dedekind zeta function of K given by

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{(\mathbf{N}\mathfrak{p})^s}\right)^{-1} \quad \text{for } \sigma > 1.$$

It is well known that the zeros of Hecke L -functions are intimately related with the distribution of prime ideals of K amongst the equivalence classes of $\text{Cl}(\mathfrak{q})$ and, by class field theory, with prime ideal decompositions in abelian extensions of K . Indeed, the analytic properties of Hecke L -functions have been widely studied by many authors such as Fogels [Fog62].

However, the known results on zeros of Hecke L -functions do not typically have explicit dependence on the field K with explicit absolute constants. In the last few years, there has been some progress in this direction with explicit results on the zero-free regions of $\zeta_K(s)$ by Kadiri [Kad12] and zero-free regions for Hecke L -functions by Ahn and Kwon [AK14]. Kadiri and Ng [KN12] have also proved a form of quantitative Deuring-Heilbronn phenomenon and explicit zero density estimates for $\zeta_K(s)$, the latter of which was subsequently improved by Trudgian [Tru15]. The aim of this paper is to provide improved and several new explicit results on the zeros of Hecke L -functions such as improved zero-free regions, the Deuring-Heilbronn phenomenon for real Hecke characters, and zero density estimates.

In the classical case $K = \mathbb{Q}$ and $\mathfrak{q} = (q)$, Hecke L -functions are the familiar Dirichlet L -functions modulo q and prime ideals in equivalence classes of $\text{Cl}(\mathfrak{q})$ naturally correspond to primes in arithmetic progressions modulo q . There is a vast literature on the zeros of Dirichlet L -functions, including many with explicit constants, and of particular importance to us is the landmark paper of Heath-Brown [HB95].

For the statement of the main theorems, note that $\nu(x)$ is *any fixed* increasing real-variable function ≥ 4 such that $\nu(x) \gg \log(x+4)$ for $x \geq 1$.

Theorem 1.1. *Suppose $d_K(\mathbf{N}\mathfrak{q})n_K^{n_K}$ is sufficiently large and let $r \geq 1$ be an integer. Then the function*

$$\prod_{\substack{\chi \pmod{\mathfrak{q}} \\ \text{ord } \chi \geq r}} L(s, \chi)$$

$$\sigma \geq 1 - \frac{c}{\log d_K + \frac{3}{4} \log \mathbf{N}\mathfrak{q} + n_K \cdot \nu(n_K)}, \quad |t| \leq 1$$

where $s = \sigma + it$ and

$$c = \begin{cases} 0.1764 & \text{if } r \geq 6, \\ 0.1489 & \text{if } r = 5, \\ 0.1227 & \text{if } r = 2, 3, 4, \\ 0.0875 & \text{if } r = 1. \end{cases}$$

Moreover, if this exceptional zero ρ_1 exists, then it and its associated character χ_1 are both real.

Corollary 1.2. *The Dedekind zeta function $\zeta_K(s)$ has at most 1 zero, counting with multiplicity, in the rectangle*

$$\sigma \geq 1 - \frac{0.0875}{\log d_K + n_K \cdot \nu(n_K)}, \quad |t| \leq 1,$$

where $s = \sigma + it$ and provided $d_K n_K^{n_K}$ is sufficiently large. If this exceptional zero exists, it is real.

Remark. For general number fields K , it is possible that the exceptional character χ_1 is principal. That is, the Dedekind zeta function $\zeta_K(s)$ may have a real zero exceptionally close to $s = 1$.

As already mentioned, some explicit results have been shown by Kadiri [Kad12] and Ahn and Kwon [AK14] for zero-free regions of $L(s, \chi)$ of the form

$$(1.1) \quad \sigma \geq 1 - \frac{c_0}{\log d_K + \log N\mathfrak{q}}, \quad |t| \leq 0.13.$$

Note that the dependence on the degree n_K has been “absorbed” into $\log d_K$. It has been shown that $L(s, \chi)$ is zero-free (except possibly for one real zero when χ is real) in the rectangle (1.1) for

$$c_0 = \begin{cases} 0.1149 & \text{if } \text{ord } \chi \geq 5 \text{ [AK14]}, \\ 0.1004 & \text{if } \text{ord } \chi = 4 \text{ [AK14]}, \\ 0.0662 & \text{if } \text{ord } \chi = 3 \text{ [AK14]}, \\ 0.0392 & \text{if } \text{ord } \chi = 2 \text{ and } d_K \text{ is sufficiently large [Kad12]}^1, \\ 0.0784 & \text{if } \text{ord } \chi = 1 \text{ and } d_K \text{ is sufficiently large [Kad12]}. \end{cases}$$

Note that the results of [Kad12] also allow for $|t| \leq 1$ to be used in (1.1). Comparing the above known values for c_0 with Theorem 1.1, if a given family of number fields K satisfies

$$(1.2) \quad n_K = O\left(\frac{\log(d_K N\mathfrak{q})}{\log \log(d_K N\mathfrak{q})}\right),$$

then, for a suitable choice of $\nu(x)$, Theorem 1.1 is superior to all previously known cases, especially in the $N\mathfrak{q}$ -aspect. A classical theorem of Minkowski states, for any number field K ,

$$n_K = O(\log d_K)$$

so, unless n_K is unusually large, one would expect that (1.2) typically holds. For example, given a fixed rational prime p , one can verify that the family of p -power cyclotomic fields $K = \mathbb{Q}(e^{2\pi i/p^m})$ satisfies (1.2).

We also establish a result, similar to those of [Gra81] and [HB95] for Dirichlet L -functions, giving a larger zero-free region but allowing more zeros.

Theorem 1.3. *Suppose $d_K(N\mathfrak{q})n_K^{n_K}$ is sufficiently large. Then $\prod_{\chi \pmod{\mathfrak{q}}} L(s, \chi)$ has at most 2 zeros, counting with multiplicity, in the rectangle*

$$\sigma \geq 1 - \frac{0.2866}{\log d_K + \frac{3}{4} \log N\mathfrak{q} + n_K \cdot \nu(n_K)} \quad |t| \leq 1.$$

¹This case is not explicitly written in the cited paper but is directly implied by the case $\text{ord } \chi = 1$.

Moreover, the Dedekind zeta function $\zeta_K(s)$ has at most 2 zeros, counting with multiplicity, in the rectangle

$$\sigma \geq 1 - \frac{0.2909}{\log d_K + n_K \cdot \nu(n_K)} \quad |t| \leq 1.$$

When an exceptional zero ρ_1 from Theorem 1.1 exists, we prove an explicit version of the well-known Deuring-Heilbronn phenomenon.

Theorem 1.4. *Let $\chi_1 \pmod{\mathfrak{q}}$ be a real character. Suppose*

$$\beta_1 = 1 - \frac{\lambda_1}{\log d_K + \frac{3}{4} \log N\mathfrak{q} + n_K \cdot \nu(n_K)}$$

is a real zero of $L(s, \chi_1)$ with $\lambda_1 > 0$. Then, provided $d_K(N\mathfrak{q})n_K^{n_K}$ is sufficiently large (depending on $R \geq 1$ and possibly $\epsilon > 0$), the function $\prod_{\chi \pmod{\mathfrak{q}}} L(s, \chi)$ has only the one zero

β_1 , counting with multiplicity, in the rectangle

$$\sigma \geq 1 - \frac{\min\{c_1 \log(1/\lambda_1), R\}}{\log d_K + \frac{3}{4} \log N\mathfrak{q} + n_K \cdot \nu(n_K)}, \quad |t| \leq 1,$$

where $s = \sigma + it$ and

$$c_1 = \begin{cases} \frac{1}{2} - \epsilon & \text{if } \chi_1 \text{ is quadratic and } \lambda_1 \leq 10^{-10}, \\ 0.2103 & \text{if } \chi_1 \text{ is quadratic and } \lambda_1 \leq 0.1227, \\ 1 - \epsilon & \text{if } \chi_1 \text{ is principal and } \lambda_1 \leq 10^{-5}, \\ 0.7399 & \text{if } \chi_1 \text{ is principal and } \lambda_1 \leq 0.0875. \end{cases}$$

In the classical case $K = \mathbb{Q}$ and $\mathfrak{q} = (q)$, Linnik [Lin44] was the pioneer of the Deuring-Heilbronn phenomenon. For other number fields, a non-explicit K -uniform variant of Theorem 1.4 is due to Lagarias-Montgomery-Odlyzko [LMO79] in the case of the Dedekind zeta function $\zeta_K(s)$ and to Weiss [Wei83] for general Hecke L -functions. Kadiri and Ng [KN12] have recently established an explicit version of the Deuring-Heilbronn phenomenon for zeros of the Dedekind zeta function $\zeta_K(s)$ with

$$c_1 = \begin{cases} 0.9045 & \text{if } \lambda_1 \leq 10^{-6}, \\ 0.6546 & \text{if } \lambda_1 \leq 0.0784. \end{cases}$$

Hence, Theorem 1.4 improves upon their result when (1.2) holds and when the primary term $c_1 \log(1/\lambda_1)$ dominates, as normally is the case.

We also establish explicit bounds related to the zero density of Hecke L -functions. For $\lambda > 0$, define $N(\lambda)$ to be the number of non-principal characters $\chi \pmod{\mathfrak{q}}$ with a zero in the region

$$(1.3) \quad \sigma \geq 1 - \frac{\lambda}{\log d_K + \frac{3}{4} \log N\mathfrak{q} + n_K \cdot \nu(n_K)}, \quad |t| \leq 1.$$

In the classical case $K = \mathbb{Q}$ and $\mathfrak{q} = (q)$, this quantity has been analyzed by [Gra81, HB95] for a slowly growing range ($\lambda \ll \log \log \log q$) and by [HB95] for a bounded range ($\lambda \leq 2$). We establish a result in the same vein as the latter. To do so, we require some technical assumptions.

Let $0 < \lambda \leq 2$ be given. Let $f \in C_c^2([0, \infty))$ have Laplace transform $F(z) = \int_0^\infty f(t)e^{-zt}dt$. Suppose f satisfies all of the following:

$$(1.4) \quad \begin{aligned} & f(t) \geq 0 \text{ for } t \geq 0; \quad \operatorname{Re}\{F(z)\} \geq 0 \text{ for } \operatorname{Re}\{z\} \geq 0; \\ & F(\lambda) > \tfrac{1}{3}f(0); \quad \left(F(\lambda) - \tfrac{1}{3}f(0)\right)^2 > \tfrac{1}{3}f(0)\left(\tfrac{1}{4}f(0) + F(0)\right). \end{aligned}$$

Then we have the following result.

Theorem 1.5. *Let $\epsilon > 0$ and $0 < \lambda \leq 2$. Suppose $f \in C_c^2([0, \infty))$ satisfies (1.4). Then unconditionally,*

$$N(\lambda) \leq \frac{\left(\tfrac{1}{4}f(0) + F(0)\right)\left(F(0) - \tfrac{1}{12}f(0)\right)}{\left(F(\lambda) - \tfrac{1}{3}f(0)\right)^2 - \tfrac{1}{3}f(0)\left(\tfrac{1}{4}f(0) + F(0)\right)} + \epsilon$$

for $d_K(N\mathfrak{q})n_K^{n_K}$ sufficiently large depending on ϵ and f .

Remark. Let ρ_1 be a certain zero of a Hecke L -function $L(s, \chi_1)$ with the property that $\operatorname{Re}\{\rho_1\} \geq \operatorname{Re}\{\rho_\chi\}$ for any zero ρ_χ in the rectangle (1.3) of any $\chi \pmod{\mathfrak{q}}$. By introducing dependence on ρ_1 , the bound on $N(\lambda)$ in Theorem 1.5 can be improved. See Section 3 for the choice of ρ_1 and Theorem 7.1 for further details.

Theorem 1.5 and its proof are inspired by [HB95, Section 12] and so similarly, the obtained bounds are non-trivial only for a narrow range of λ . By choosing² f roughly optimally, we exhibit a table of bounds derived from Theorem 1.5 below.

λ	.100	.125	.150	.175	.200	.225	.250	.275	.300	.325	.350	.375	.400	.425
$N(\lambda)$	2	2	3	3	4	4	5	6	7	9	11	15	22	46

One can see that the estimates obtained are comparable to Theorems 1.1 and 1.3 which respectively imply that $N(0.1227) \leq 1$ and $N(0.2866) \leq 2$.

In the classical case $K = \mathbb{Q}$, Heath-Brown substantially improved upon all preceding work for zeros of Dirichlet L -functions and so, for general number fields K , we have taken advantage of the innovations founded in [HB95] to improve on the existing aforementioned results and also to establish new explicit estimates. As such, the general structure of this paper is reminiscent of his work and is subject to small improvements similar to those suggested in [HB95, Section 16]. Xylouris implemented a number of those suggestions in [Xyl11] so in principle one could refine the results here by the same methods.

Finally, we describe the organization of the paper. Section 2 covers well-known facts about Hecke L -functions and some elementary estimates. In Section 3, we specify some frequently-used notation and identify zeros of Hecke L -functions which will play a key role throughout the paper. Sections 4, 5 and 6 establish several different “explicit inequalities” related to $-\frac{L'}{L}(s, \chi)$ by involving classical arguments, higher derivatives of $-\frac{L'}{L}(s, \chi)$, and smooth weights. The results therein form the technical crux of all subsequent proofs and applications. Section 7 provides bounds for the zero density quantity $N(\lambda)$. Section 8 quantifies Deuring-Heilbronn phenomenon for the exceptional case. Section 9 deals with the

²See the discussion following the proof of Theorem 7.1 for details.

milder zero repulsion in the non-exceptional case. Section 10 establishes a zero-free region for Hecke L -functions.

For the reader who wishes to proceed quickly to the proofs of the theorems:

- Theorem 1.1 and Corollary 1.2 are proved in Section 10.
- Theorem 1.3 is an immediate corollary of Propositions 8.7, 8.13, 9.4 and 9.10.
- Theorem 1.4 is an immediate corollary of Propositions 8.7 and 8.13.
- Theorem 1.5 is a special case of Theorem 7.1.

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2. PRELIMINARIES

2.1. Hecke L -functions. Recall *Hecke characters* are characters χ of the ray class group $\text{Cl}(\mathfrak{q}) = I(\mathfrak{q})/P_{\mathfrak{q}}$. We often write $\chi(\bmod \mathfrak{q})$ to indicate this relationship. For notational convenience, we pullback the domain of χ to $I(\mathfrak{q})$ and then extend it to all of $I(\mathcal{O})$ by zero; that is, $\chi(\mathfrak{n})$ is defined for all integral ideals $\mathfrak{n} \subseteq \mathcal{O}$ and $\chi(\mathfrak{n}) = 0$ for $(\mathfrak{n}, \mathfrak{q}) \neq 1$. The *conductor* \mathfrak{f}_{χ} of a Hecke character $\chi(\bmod \mathfrak{q})$ is the maximal integral ideal such that χ is the push-forward of a Hecke character modulo \mathfrak{f}_{χ} . It follows that \mathfrak{f}_{χ} divides \mathfrak{q} . We say χ is *primitive modulo* \mathfrak{q} if $\mathfrak{f}_{\chi} = \mathfrak{q}$.

Thus, the Hecke L -function associated to $\chi(\bmod \mathfrak{q})$ may be written as

$$L(s, \chi) = \sum_{\mathfrak{n} \subseteq \mathcal{O}} \chi(\mathfrak{n})(N\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{(N\mathfrak{p})^s}\right)^{-1} \quad \text{for } \sigma > 1$$

where $s = \sigma + it \in \mathbb{C}$. Unless otherwise specified, we shall henceforth refer to Hecke characters as characters.

Functional Equation. Let $\chi(\bmod \mathfrak{f}_{\chi})$ be a primitive character. Recall that the *L -function of χ at infinity* is given by

$$(2.1) \quad L_{\infty}(s, \chi) := \left[\pi^{-s/2} \Gamma\left(\frac{s}{2}\right)\right]^{a(\chi)} \cdot \left[\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right)\right]^{b(\chi)}$$

where $\Gamma(s)$ is the Gamma function and $a(\chi), b(\chi)$ are certain non-negative integers satisfying

$$a(\chi) + b(\chi) = [K : \mathbb{Q}] = n_K.$$

Then the *completed L -function of $L(s, \chi)$* is defined to be

$$(2.2) \quad \xi(s, \chi) := \begin{cases} (d_K N\mathfrak{f}_{\chi})^{s/2} L(s, \chi) L_{\infty}(s, \chi) & \text{if } \chi \neq \chi_0, \\ (d_K N\mathfrak{f}_{\chi})^{s/2} L(s, \chi) L_{\infty}(s, \chi) \cdot s(1-s) & \text{if } \chi = \chi_0. \end{cases}$$

With an appropriate choice of $a(\chi)$ and $b(\chi)$, it is well-known that $\xi(s, \chi)$ is an entire function satisfying the functional equation

$$(2.3) \quad \xi(s, \chi) = \varepsilon(\chi) \cdot \xi(1-s, \overline{\chi})$$

where $\varepsilon(\chi) \in \mathbb{C}$ is the global root number having absolute value 1. See [LO77, Section 5] for details.

Convexity Bound.

Lemma 2.1. *Let $\delta \in (0, \frac{1}{2})$ be given. Suppose χ is a primitive non-principal Hecke character modulo \mathfrak{f}_χ . Then*

$$|L(s, \chi)| \ll \zeta_{\mathbb{Q}}(1 + \delta)^{n_K} \left(\frac{d_K N \mathfrak{f}_\chi}{(2\pi)^{n_K}} (1 + |s|)^{n_K} \right)^{(1-\sigma+\delta)/2}$$

and

$$|(s-1) \cdot \zeta_K(s)| \ll \zeta_{\mathbb{Q}}(1 + \delta)^{n_K} \left(\frac{d_K}{(2\pi)^{n_K}} (1 + |s|)^{n_K} \right)^{(1-\sigma+\delta)/2}$$

uniformly in the region

$$-\delta \leq \sigma \leq 1 + \delta.$$

Proof. This is a version of [Rad60, Theorem 5] which has been simplified for our purposes. In his notation, the constants v_q, a_p, a_{p+r_2}, v_p are all zero for characters of $\text{Cl}(\mathfrak{q})$. Recall that $\zeta_{\mathbb{Q}}(\cdot)$ is the classical Riemann zeta function. \square

Explicit Formula. Using the Hadamard product for $\xi(s, \chi)$, one may derive an explicit formula for the logarithmic derivative of $L(s, \chi)$. Before recording this classical result, we introduce an additional piece of notation which will be used throughout the paper:

$$(2.4) \quad E_0(\chi) := \begin{cases} 1 & \text{if } \chi \text{ is principal,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Let χ be a primitive Hecke character modulo \mathfrak{f}_χ . Then for all $s \in \mathbb{C}$ away from zeros of $\xi(s, \chi)$,*

$$-\frac{L'}{L}(s, \chi) = \frac{E_0(\chi)}{s-1} + \frac{E_0(\chi)}{s} + \frac{1}{2} \log(d_K N \mathfrak{q}) + \frac{L'_\infty}{L_\infty}(s, \chi) - B(\chi) - \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right),$$

where $B(\chi) \in \mathbb{C}$ is a constant depending on χ and the conditionally convergent sum is over all zeros ρ of $\xi(s, \chi)$. Moreover,

$$\text{Re}\{B(\chi)\} = -\frac{1}{2} \sum_{\rho} \left(\frac{1}{1-\rho} + \frac{1}{\rho} \right) = -\sum_{\rho} \text{Re} \frac{1}{\rho} < 0.$$

Proof. See [LO77, Section 5] for a proof. Note “ \sum_{ρ} ” denotes “ $\lim_{T \rightarrow \infty} \sum_{|\text{Im} \rho| \leq T}$ ”. \square

Lemma 2.2 gives the desired formula for $-\frac{L'}{L}(s, \chi)$ with only $\frac{L'_\infty}{L_\infty}(s, \chi)$ to be estimated.

Lemma 2.3. *Let χ be a primitive Hecke character. If $\text{Re}\{s\} \geq 1/8$, then*

$$\frac{L'_\infty}{L_\infty}(s, \chi) \ll n_K \log(2 + |s|).$$

Proof. See [LO77, Lemma 5.3]. \square

2.2. Elementary Estimates.

Lemma 2.4. *Let \mathfrak{q} be an integral ideal. Then, for $\epsilon > 0$,*

$$\sum_{\mathfrak{p}|\mathfrak{q}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}} \leq \sqrt{n_K \log N\mathfrak{q}} \leq \frac{1}{2} \left(\frac{n_K}{\epsilon} + \epsilon \log N\mathfrak{q} \right)$$

where the sum is over prime ideals \mathfrak{p} dividing \mathfrak{q} .

Proof. The second inequality follows from $(x+y)/2 \geq \sqrt{xy}$ for $x, y \geq 0$. It suffices to prove the first estimate. Write $\mathfrak{q} = \prod_{i=1}^r \mathfrak{p}_i^{e_i}$ in its unique ideal factorization where \mathfrak{p}_i are distinct prime ideals and $e_i \geq 1$. Denote $q_i = N\mathfrak{p}_i$ and $a_m = \#\{i : q_i = m\}$. Observe that $a_m = 0$ unless m is a power of a rational prime p . Since the principal ideal (p) factors into at most n_K prime ideals in K , it follows $a_m \leq n_K$ for $m \geq 1$. Thus, by Cauchy-Schwarz,

$$\begin{aligned} \sum_{\mathfrak{p}|\mathfrak{q}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}} &= \sum_{i=1}^r \frac{\log q_i}{q_i} \leq \left(\sum_{i=1}^r \frac{\log q_i}{q_i^2} \right)^{1/2} \left(\sum_{i=1}^r \log q_i \right)^{1/2} \\ &= \left(\sum_{m \geq 1} a_m \frac{\log m}{m^2} \right)^{1/2} \left(\sum_{i=1}^r \log q_i \right)^{1/2} \\ &\leq n_K^{1/2} \left(\sum_{m \geq 1} \frac{\log m}{m^2} \right)^{1/2} \left(\sum_{i=1}^r e_i \log q_i \right)^{1/2} \\ &= \left(\sum_{m \geq 1} \frac{\log m}{m^2} \right)^{1/2} \sqrt{n_K \log N\mathfrak{q}} \end{aligned}$$

Since $\sum_{m \geq 1} \frac{\log m}{m^2} < 1$, the result follows. \square

Lemma 2.5. *For $\sigma > 1$,*

$$\begin{aligned} \zeta_K(\sigma) &\leq \zeta(\sigma)^{n_K} \leq \left(\frac{\sigma}{\sigma-1} \right)^{n_K}, \\ \log \zeta_K(\sigma) &\leq n_K \log \left(\frac{\sigma}{\sigma-1} \right), \\ -\frac{\zeta'_K(\sigma)}{\zeta_K(\sigma)} &\leq -n_K \frac{\zeta'(\sigma)}{\zeta(\sigma)} \leq \frac{n_K}{\sigma-1}. \end{aligned}$$

where $\zeta(s) = \zeta_{\mathbb{Q}}(s)$ is the classical Riemann zeta function.

Proof. For the first inequality, observe

$$\zeta_K(\sigma) = \prod_{\mathfrak{p}} (1 - (N\mathfrak{p})^{-\sigma})^{-1} = \prod_p \prod_{(p) \subseteq \mathfrak{p}} (1 - (N\mathfrak{p})^{-\sigma})^{-1} \leq \prod_p (1 - p^{-\sigma})^{-n_K} = \zeta(\sigma)^{n_K}$$

and note $\zeta(\sigma) \leq \left(\frac{\sigma}{\sigma-1} \right)$ from [MV07, Corollary 1.14]. The second inequality follows easily from the first. The third inequality follows by an argument similar to that of the first and additionally noting $-\frac{\zeta'}{\zeta}(\sigma) < \frac{1}{\sigma-1}$ by [Lou92, Lemma (a)] for example. \square

Lemma 2.6. *Let $k \geq 1$ and $\chi \pmod{\mathfrak{q}}$ be a Hecke character. Then*

$$\frac{1}{k!} \frac{d^k}{ds^k} \frac{L'_{\infty}(s, \chi)}{L_{\infty}} \ll n_K$$

provided $\operatorname{Re}\{s\} > 1$.

Proof. Denote $\psi^{(k)}(\cdot) = \frac{d^k}{ds^k} \frac{\Gamma'}{\Gamma}(\cdot)$. From (2.1), we have that

$$(2.5) \quad \frac{d^k}{ds^k} \frac{L'_\infty}{L_\infty}(s, \chi) = \frac{a(\chi)}{2^{k+1}} \cdot \psi^{(k)}\left(\frac{s}{2}\right) + \frac{b(\chi)}{2^{k+1}} \cdot \psi^{(k)}\left(\frac{s+1}{2}\right).$$

Since $a(\chi) + b(\chi) = n_K$, it suffices to bound $\psi^{(k)}(z)$ for $\operatorname{Re}\{z\} > 1/2$. From the well-known logarithmic derivative of the Gamma function (see [MV07, (C.10)] for example), observe

$$\left| \frac{\psi^{(k)}(z)}{k!} \right| = \left| (-1)^k \sum_{n=1}^{\infty} \frac{1}{(n+z)^{k+1}} \right| \leq 2^{k+1} \sum_{n \text{ odd}} \frac{1}{n^{k+1}} = (2^{k+1} - 1) \zeta_{\mathbb{Q}}(k+1)$$

for $\operatorname{Re}\{z\} > 1/2$, which yields the desired result when combined with (2.5) as $\zeta_{\mathbb{Q}}(k+1) \leq \zeta_{\mathbb{Q}}(2) = \pi^2/6$. \square

3. ZERO-FREE GAP AND LABELLING OF ZEROS

The main goal of this section is to show that there is a thin rectangle inside the critical strip above which there is a zero-free gap for

$$Z(s) := \prod_{\chi \pmod{\mathfrak{q}}} L(s, \chi).$$

Afterwards, we label important zeros of $Z(s)$ which we will refer to throughout the paper. This zero-free gap is necessary for the proof of Lemma 6.3 – a crucial component for later sections.

Let $\vartheta \in [\frac{3}{4}, 1]$ be fixed³ and let $\nu(x), \eta(x)$ be any fixed increasing functions for $x \in [1, \infty)$ such that

$$(3.1) \quad \begin{aligned} \nu(x) &\in [4, \infty), & \nu(x) &\gg \log(x+4), \\ \eta(x) &\in [2, \infty), & \eta(x) &\rightarrow \infty \text{ as } x \rightarrow \infty, \end{aligned} \quad \text{and } \frac{x}{\eta(x) \log(x+1)} \text{ is increasing.}$$

One could take $\eta(x) = \frac{1}{2} \log x + 2$, for example. Denote

$$(3.2) \quad \begin{aligned} \mathcal{L} &:= \log d_K + \vartheta \cdot \log N\mathfrak{q} + n_K \cdot \nu(n_K), \\ \mathcal{L}^* &:= \log d_K + \vartheta \cdot \log N\mathfrak{q}, \\ \mathcal{T} &:= (\mathcal{L}^*)^{1/\eta(n_K) \log(n_K+1)} + \nu(n_K). \end{aligned}$$

Similarly, for a Hecke character $\chi \pmod{\mathfrak{q}}$ with conductor \mathfrak{f}_χ , define

$$(3.3) \quad \begin{aligned} \mathcal{L}_\chi &:= \log d_K + \log N\mathfrak{f}_\chi + n_K \cdot \nu(n_K), \\ \mathcal{L}_\chi^* &:= \log d_K + \log N\mathfrak{f}_\chi \\ \mathcal{L}_0 &:= \mathcal{L}_{\chi_0} = \log d_K + n_K \cdot \nu(n_K) \end{aligned}$$

For the remainder of the paper, we shall maintain this notation because these quantities will be ubiquitous in all of our estimates. *Henceforth, all implicit constants will be absolute (in particular, independent of K, \mathfrak{q} and all Hecke characters χ modulo \mathfrak{q}) and will only implicitly*

³For the purposes of this paper, setting $\vartheta = \frac{3}{4}$ would be sufficient but we wish to maintain flexibility for possible future investigations of Hecke L -functions.

depend on the fixed ϑ, ν and η .

First, we record some simple relationships between the quantities defined in (3.2) and (3.3).

Lemma 3.1. *For the quantities defined in (3.2) and (3.3), all of the following hold:*

- (i) $4 \leq \mathcal{T} \leq \mathcal{L}$.
- (ii) $n_K \log \mathcal{T} = o(\mathcal{L})$.
- (iii) $\mathcal{L}^* + n_K \log \mathcal{T} \leq \mathcal{L} + o(\mathcal{L})$ and $\mathcal{L}_\chi^* + n_K \log \mathcal{T} \leq \mathcal{L}_\chi + o(\mathcal{L})$.
- (iv) $\mathcal{T} \rightarrow \infty$ if and only if $\mathcal{L} \rightarrow \infty$.
- (v) $a\mathcal{L}_0 + b\mathcal{L}_\chi \leq (a+b)\mathcal{L}$ provided $0 \leq b \leq 3a$.

Proof. Statements (i) and (iii) follow easily from (ii) and/or the definitions of \mathcal{T}, \mathcal{L} and \mathcal{L}^* . For (ii), observe that

$$n_K \log \mathcal{T} \leq \frac{n_K \log \mathcal{L}^*}{\eta(n_K) \log(n_K + 1)} + n_K \log \nu(n_K) + n_K \log 2$$

The second and third terms are $o(\mathcal{L})$ as $\nu(x)$ is increasing. For the first term, note that $\frac{n_K}{\eta(n_K) \log(n_K + 1)}$ is increasing as a function n_K by (3.1) and so plugging in the upper bound $n_K = O(\log d_K) = O(\mathcal{L}^*)$ from Minkowski's theorem we deduce

$$n_K \log \mathcal{T} \ll \frac{\mathcal{L}^*}{\eta(\mathcal{L}^*)} + o(\mathcal{L}) = o(\mathcal{L})$$

since $\eta(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $\mathcal{L}^* \leq \mathcal{L}$. For (iv), if n_K is bounded, then necessarily $\mathcal{L}^* \rightarrow \infty$ in which case both \mathcal{T} and \mathcal{L} approach infinity. Otherwise, if $n_K \rightarrow \infty$, then both \mathcal{T} and \mathcal{L} approach infinity since $\nu(x) \rightarrow \infty$ as $x \rightarrow \infty$. For (v), the claim follows from the fact that $\vartheta \geq \frac{3}{4}$. \square

Next, we establish the desired zero-free gap which motivates the choice of \mathcal{L} and its related quantities.

Lemma 3.2. *Let $C_0 > 0$ be a sufficiently large absolute constant and let \mathcal{T} be defined as in (3.2). For \mathcal{L} sufficiently large, there exists a positive integer $T_0 = T_0(\mathfrak{q}) \leq \frac{\mathcal{T}}{10}$ such that $\prod_{\chi \pmod{\mathfrak{q}}} L(s, \chi)$ has no zeros in the region*

$$1 - \frac{\log \log \mathcal{T}}{C_0 \mathcal{L}} \leq \sigma \leq 1, \quad T_0 \leq |t| \leq 10T_0.$$

Proof. For $0 \leq \alpha \leq 1$ and $T \geq 0$, denote

$$N_{\mathfrak{q}}(\alpha, T) = \sum_{\chi \pmod{\mathfrak{q}}} \#\{\rho \in \mathbb{C} \mid L(\rho, \chi) = 0, \quad \alpha \leq \beta \leq 1, \quad 0 \leq |\gamma| \leq T\}$$

where we count zeros with multiplicity. We shall apply a simplified version of a result of Weiss [Wei83, Theorem 4.3]; in his notation, we restrict to the case $H = P_{\mathfrak{q}}$ and $Q = n_K^{n_K} d_K N_{\mathfrak{q}}$ from which it follows $h_H = \#I(\mathfrak{q})/H \leq e^{O(\mathcal{L})}$ by [Wei83, Lemma 1.16] and also $Q \leq e^{O(\mathcal{L})}$ since $\nu(x) \gg \log(x+4)$ by (3.1). Therefore, by [Wei83, Theorem 4.3], for $c_6 \leq \alpha \leq 1 - \frac{c_7}{\mathcal{L}}$, we have

$$(3.4) \quad N_{\mathfrak{q}}(\alpha, T) \ll (e^{\mathcal{L}} \cdot T^{n_K})^{c_8(1-\alpha)}$$

for some absolute constants $0 < c_6 < 1, c_7 > 0, c_8 > 0$ and provided T and \mathcal{L} are sufficiently large. Now suppose, for a contradiction, that no such T_0 exists. Setting $\alpha = 1 - \frac{\log \log \mathcal{T}}{C_0 \mathcal{L}}$, it follows that every region

$$\alpha \leq \sigma \leq 1, \quad 10^j \leq |t| \leq 10^{j+1},$$

for $0 \leq j < J$ where $J := \left\lceil \frac{\log \mathcal{T}}{\log 10} \right\rceil$, contains at least one zero of $\prod_{\chi \pmod{\mathfrak{q}}} L(s, \chi)$. Hence,

$$N_{\mathfrak{q}}(\alpha, \mathcal{T}) \geq J \gg \log \mathcal{T}.$$

On the other hand, by (3.4) with $T = \mathcal{T}$ sufficiently large, our choice of α implies

$$N_{\mathfrak{q}}(\alpha, \mathcal{T}) \ll \left(e^{4\mathcal{L}/3} \mathcal{T}^{n_K} \right)^{c_8(1-\alpha)} \ll \exp \left(\frac{c_8}{C_0} \left(\frac{4}{3} \log \log \mathcal{T} + \frac{n_K \log \mathcal{T} \log \log \mathcal{T}}{\mathcal{L}} \right) \right).$$

From Lemma 3.1, $n_K \log \mathcal{T} = o(\mathcal{L})$ so for some absolute constant $c_9 > 0$

$$N_{\mathfrak{q}}(\alpha, \mathcal{T}) \ll \exp \left(\frac{c_9}{C_0} \log \log \mathcal{T} \right) \ll (\log \mathcal{T})^{\frac{c_9}{C_0}}.$$

Upon taking $C_0 = 2c_9$, we obtain a contradiction for \mathcal{T} sufficiently large. From Lemma 3.1, we may equivalently ask that \mathcal{L} is sufficiently large. \square

Using the zero-free gap from Lemma 3.2, we label important “bad” zeros of $Z(s) = \prod_{\chi \pmod{\mathfrak{q}}} L(s, \chi)$. These zeros will be referred to throughout the paper. A typical zero of $L(s, \chi)$ will be denoted $\rho = \beta + i\gamma$ or $\rho_{\chi} = \beta_{\chi} + i\gamma_{\chi}$ when necessary.

Worst Zero of each Character
Consider the rectangle

$$\mathcal{R} = \mathcal{R}(\mathfrak{q}) := \{s \in \mathbb{C} : 1 - \frac{\log \log \mathcal{T}}{C_0 \mathcal{L}} \leq \sigma \leq 1, \quad |t| \leq T_0\}$$

for $T_0 = T_0(\mathfrak{q}) \in [1, \frac{\mathcal{T}}{10}]$ and $C_0 > 0$ defined by Lemma 3.2. Denote \mathcal{Z} to be the multiset of zeros of $Z(s)$ contained in \mathcal{R} . Choose finitely many zeros ρ_1, ρ_2, \dots from \mathcal{Z} as follows:

- (1) Pick ρ_1 such that β_1 is maximal, and let χ_1 be the corresponding character. Remove all zeros of $L(s, \chi_1)$ and $L(s, \overline{\chi_1})$ from \mathcal{Z} .
- (2) Pick ρ_2 such that β_2 is maximal, and let χ_2 be the corresponding character. Remove all zeros of $L(s, \chi_2)$ and $L(s, \overline{\chi_2})$ from \mathcal{Z} .
- \vdots

Continue in this fashion until \mathcal{R} has no more zeros to choose. Then if $\chi \neq \chi_i, \overline{\chi_i}$ for $1 \leq i < k$, then by Lemma 3.2 every zero ρ of $L(s, \chi)$ satisfies:

$$(3.5) \quad \operatorname{Re}(\rho) \leq \operatorname{Re}(\rho_k) \quad \text{or} \quad |\operatorname{Im}(\rho)| \geq 10T_0.$$

For convenience of notation, denote

$$\rho_k = \beta_k + i\gamma_k, \quad \beta_k = 1 - \frac{\lambda_k}{\mathcal{L}}, \quad \gamma_k = \frac{\mu_k}{\mathcal{L}}.$$

Second Worst Zero of the Worst Character

Suppose $L(s, \chi_1)$ has a zero $\rho' \neq \rho_1, \overline{\rho_1}$ in the rectangle \mathcal{R} , or possibly a repeated real zero $\rho' = \rho_1$. Choose ρ' with $\text{Re}(\rho')$ maximal and write

$$\rho' = \beta' + i\gamma', \quad \beta' = 1 - \frac{\lambda'}{\mathcal{L}}, \quad \gamma' = \frac{\mu'}{\mathcal{L}}.$$

4. CLASSICAL EXPLICIT INEQUALITY

In this section, we prove an inequality for $-\text{Re}\{\frac{L'}{L}(s, \chi)\}$ based on a bound for $L(s, \chi)$ in the critical strip and a type of Jensen's formula employed by Heath-Brown in [HB95, Section 3]. First, we deal with non-primitive characters.

Lemma 4.1. *Suppose $\chi \pmod{\mathfrak{q}}$ is induced by the character $\chi^* \pmod{\mathfrak{f}}$. Then for $\epsilon > 0$, we have*

$$\frac{L'}{L}(s, \chi) = \frac{L'}{L}(s, \chi^*) + O\left(\frac{n_K}{\epsilon} + \epsilon \mathcal{L}^*\right)$$

uniformly in the range $\sigma > 1$.

Proof. Observe

$$\left| \frac{L'}{L}(s, \chi) - \frac{L'}{L}(s, \chi^*) \right| \leq \sum_{\mathfrak{p}|\mathfrak{q}} \sum_{j \geq 1} \frac{\log N\mathfrak{p}}{(N\mathfrak{p})^j} \leq 2 \sum_{\mathfrak{p}|\mathfrak{q}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}}.$$

The desired result then follows from Lemma 2.4. \square

Next, we give a bound for $L(s, \chi)$ in a relevant region of the critical strip.

Lemma 4.2. *Let $\chi \pmod{\mathfrak{q}}$ be non-principal and induced by the primitive character $\chi^* \pmod{\mathfrak{f}_\chi}$. There exists an absolute constant $\phi > 0$ such that for $\epsilon > 0$, we have*

$$|L(s, \chi^*)| \leq \exp \left\{ 2\phi \mathcal{L}_\chi (1 - \sigma + \epsilon) + o_\epsilon(\mathcal{L}) \right\}$$

and

$$|(s-1) \cdot \zeta_K(s)| \leq \exp \left\{ 2\phi \mathcal{L}_0 (1 - \sigma + \epsilon) + o_\epsilon(\mathcal{L}) \right\}$$

uniformly in the region

$$\frac{1}{2} \leq \sigma \leq 1 + \frac{\log \mathcal{L}}{\mathcal{L}}, \quad |t| \leq \mathcal{T}.$$

In particular, we may take $\phi = \frac{1}{4}$.

Remark. The term $o_\epsilon(\mathcal{L})/\mathcal{L}$ goes to zero as $\mathcal{L} \rightarrow \infty$ but the rate of convergence depends on $\epsilon > 0$.

Proof. Take $\phi = \frac{1}{4}$. From Lemma 2.1, we have

$$L(s, \chi^*) \ll \zeta(1 + \epsilon)^{n_K} \left(\frac{d_K N\mathfrak{f}_\chi}{(2\pi)^{n_K}} (1 + |s|)^{n_K} \right)^{2\phi(1 - \sigma + \epsilon)}$$

uniformly in the region $-\epsilon \leq \sigma \leq 1 + \epsilon$. Noting $1 + |s| \leq 3 + \mathcal{T}$ and using Lemma 2.5, the above is therefore

$$\ll \exp \left\{ (\mathcal{L}_\chi^* + n_K \log(\mathcal{T} + 3)) \cdot 2\phi(1 - \sigma + \epsilon) + O_\epsilon(n_K) \right\}$$

From Lemma 3.1, it follows $\mathcal{L}_\chi^* + n_K \log(\mathcal{T} + 3) \leq \mathcal{L}_\chi + o(\mathcal{L})$. Substituting this into the above, the desired result then follows upon noting $n_K = o(\mathcal{L})$. The proof for $\zeta_K(s)$ is similar. \square

Any improvement on the constant ϕ will have a crucial effect on the final result. For example, the Lindelöf hypothesis for Hecke L -functions gives $\phi = \epsilon$. For the remainder of this paper, we set

$$\phi := \frac{1}{4}.$$

We may now establish the main result of this section.

Lemma 4.3. *Let $\chi \pmod{\mathfrak{q}}$ be arbitrary. For any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that*

$$-\operatorname{Re} \frac{L'}{L}(s, \chi) \leq (\phi + \epsilon) \mathcal{L}_\chi + \operatorname{Re} \left\{ \frac{E_0(\chi)}{s-1} \right\} - \sum_{|1+it-\rho| \leq \delta} \operatorname{Re} \left\{ \frac{1}{s-\rho} \right\} + o_\epsilon(\mathcal{L})$$

uniformly for

$$1 + \frac{1}{\mathcal{L} \log \mathcal{L}} \leq \sigma \leq 1 + \frac{\log \mathcal{L}}{\mathcal{L}}, \quad |t| \leq \mathcal{T}.$$

Remark. When $K = \mathbb{Q}$, Heath-Brown showed the same inequality in [HB95, Lemma 3.1] with $\phi = \frac{1}{6}$ instead of $\phi = \frac{1}{4}$ by leveraging Burgess' estimate for character sums. Note that Heath-Brown's notation for ϕ differs by a factor of 2 with our notation.

Proof. We closely follow the arguments leading to the proof of [HB95, Lemma 3.1]. Assume without loss that $\epsilon \in (0, 1/2)$. Suppose $\chi \pmod{\mathfrak{q}}$ is induced from a non-principal primitive character $\chi^* \pmod{\mathfrak{f}}$. Apply [HB95, Lemma 3.2] with $f(\cdot) = L(\cdot, \chi^*)$ with $a = s$ and $R = \frac{1}{2}$. Then

$$(4.1) \quad -\operatorname{Re} \frac{L'}{L}(s, \chi^*) = - \sum_{|s-\rho| < \frac{1}{2}} \operatorname{Re} \left\{ \frac{1}{s-\rho} - 4(s-\rho) \right\} - J$$

where

$$J := \frac{2}{\pi} \int_0^{2\pi} (\cos \theta) \cdot \log |L(s + \frac{1}{2}e^{i\theta}, \chi^*)| d\theta.$$

We require a lower bound for J so we divide the contribution of the integral into three separate intervals depending on the sign of $\cos \theta$.

- For $\theta \in [0, \pi/2]$, by Lemma 2.5 it follows that

$$\log |L(s + \frac{1}{2}e^{i\theta}, \chi^*)| \leq \log \zeta_K(\sigma + \frac{1}{2} \cos \theta) \leq n_K \log \left(\frac{2}{\sigma - 1 + \frac{1}{2} \cos \theta} \right)$$

On the interval $I_1 := [0, \frac{\pi}{2} - (\sigma - 1)]$, as $\sigma - 1 \geq 0$, the contribution of the integral J is

$$\ll n_K \int_0^{\pi/2} (\cos \theta) \log(4/\cos \theta) d\theta \ll n_K.$$

On the interval $I_2 := [\frac{\pi}{2} - (\sigma - 1), \frac{\pi}{2}]$, as $\cos \theta \geq 0$, the contribution of the integral J is

$$\ll n_K \log(2/(\sigma - 1)) \int_{I_2} (\cos \theta) d\theta \ll n_K (\sigma - 1) \log(2/(\sigma - 1)) \ll \frac{n_K (\log \mathcal{L})^2}{\mathcal{L}} \ll (\log \mathcal{L})^2$$

because $(\mathcal{L} \log \mathcal{L})^{-1} \leq \sigma - 1 \leq \mathcal{L}^{-1} \log \mathcal{L}$ and $n_K \ll \mathcal{L}$.

- For $\theta \in [\pi/2, 3\pi/2]$, notice

$$\frac{1}{2} \leq \sigma - \frac{1}{2} \leq \sigma + \frac{1}{2} \cos \theta \leq \sigma \leq 1 + \frac{\log \mathcal{L}}{\mathcal{L}}.$$

Hence, we may use Lemma 4.2 to see that

$$\begin{aligned} \log |L(s + \tfrac{1}{2}e^{i\theta}, \chi^*)| &\leq 2\phi \mathcal{L}_\chi (1 - \sigma - \tfrac{1}{2} \cos \theta + \epsilon) + o_\epsilon(\mathcal{L}) \\ &\leq 2\phi \mathcal{L}_\chi (-\tfrac{1}{2} \cos \theta + \epsilon) + o_\epsilon(\mathcal{L}). \end{aligned}$$

This implies that

$$\begin{aligned} \int_{\pi/2}^{3\pi/2} (\cos \theta) \cdot \log |L(s + \tfrac{1}{2}e^{i\theta}, \chi)| d\theta &\geq 2\phi \mathcal{L}_\chi \int_{\pi/2}^{3\pi/2} (-\tfrac{1}{2} \cos^2 \theta + \epsilon \cos \theta) d\theta + o_\epsilon(\mathcal{L}) \\ &= \phi \mathcal{L}_\chi (-\tfrac{\pi}{2} - 4\epsilon) + o_\epsilon(\mathcal{L}). \end{aligned}$$

- For $\theta \in [3\pi/2, 2\pi]$, we obtain the same contribution as $\theta \in [0, \pi/2]$ by a similar argument.

Combining all contributions, we have

$$(4.2) \quad J \geq \left(-\phi - \frac{2\epsilon}{\pi}\right) \mathcal{L}_\chi + o_\epsilon(\mathcal{L})$$

since $n_K = o(\mathcal{L})$. For the sum over zeros in (4.1), notice that we may arbitrarily discard zeros from the sum since for $|s - \rho| < \frac{1}{2}$,

$$\operatorname{Re}\left\{\frac{1}{s - \rho} - 4(s - \rho)\right\} = (\sigma - \beta) \left(\frac{1}{|s - \rho|^2} - \frac{1}{(\frac{1}{2})^2}\right) \geq 0.$$

Thus, for any $0 < \delta < \frac{1}{2} - \frac{\log \mathcal{L}}{\mathcal{L}}$, we may restrict our sum over zeros from $|s - \rho| < \frac{1}{2}$ to a smaller circle within it: $|1 + it - \rho| \leq \delta$. From our previous observation, we may discard zeros outside of this smaller circle. As $\mathcal{L} \geq 4$ by Lemma 3.1, we may instead impose that $0 < \delta < \frac{1}{8}$.

Now, from [LMO79, Lemma 2.1] and Lemma 3.1, we have that

$$\#\{\rho : |1 + it - \rho| \leq \delta\} \ll \mathcal{L}_\chi^* + n_K \log(\mathcal{T} + 3) \ll \mathcal{L}_\chi + o(\mathcal{L}).$$

Further, for such zeros ρ satisfying $|1 + it - \rho| \leq \delta$, notice

$$\operatorname{Re}\{s - \rho\} = \sigma - \beta \leq \frac{\log \mathcal{L}}{\mathcal{L}} + \delta$$

implying, for some absolute constant $c_0 \geq 1$,

$$(4.3) \quad \sum_{|1+it-\rho|\leq\delta} \operatorname{Re}\{4(s - \rho)\} \leq 4c_0(\mathcal{L}_\chi + o(\mathcal{L})) \left(\frac{\log \mathcal{L}}{\mathcal{L}} + \delta\right) = 4c_0\delta \mathcal{L}_\chi + O(\log \mathcal{L}).$$

Combining these observations, from (4.1), (4.2), and (4.3) and Lemma 4.1 we see that

$$\begin{aligned} -\operatorname{Re} \frac{L'}{L}(s, \chi) &\leq - \sum_{|1+it-\rho|\leq\delta} \operatorname{Re}\left\{\frac{1}{s - \rho} - 4(s - \rho)\right\} + \left(\phi + \frac{2\epsilon}{\pi}\right) \mathcal{L}_\chi + o_\epsilon(\mathcal{L}) \\ &\leq - \sum_{|1+it-\rho|\leq\delta} \operatorname{Re}\left\{\frac{1}{s - \rho}\right\} + \left(\phi + \frac{2\epsilon}{\pi} + 4c_0\delta\right) \mathcal{L}_\chi + o_\epsilon(\mathcal{L}). \end{aligned}$$

Taking $\delta = \epsilon/4c_0$, note $\delta \in (0, 1/8)$ as $\epsilon < 1/2$ by assumption and $c_0 \geq 1$. Rescaling ϵ appropriately completes the proof for χ non-principal.

If $\chi = \chi_0$ is principal, then we proceed in the same manner, applying Lemma 3.2 of [HB95] with $f(\cdot) = (s-1)\zeta_K(\cdot)$. This choice gives rise to the additional term $\frac{1}{s-1}$, but otherwise we continue with the same argument. The only other difference occurs in the analysis of the integral J for $\theta \in [0, \pi/2] \cup [3\pi/2, 2\pi]$ where we must instead estimate

$$\log \left(\left| s - 1 + \frac{1}{2}e^{i\theta} \right| \cdot \left| \zeta_K(s + \frac{1}{2}e^{i\theta}) \right| \right).$$

As $|s - 1 + \frac{1}{2}e^{i\theta}| \geq \frac{1}{2}$, the contribution to the integral J will be bounded and hence ignored. \square

5. POLYNOMIAL EXPLICIT INEQUALITY

By including higher derivatives of $-\frac{L'}{L}(s, \chi)$, the goal of this section is establish a generalization of the “classical explicit inequality” based on techniques in [HB95, Section 4]. Let $\chi \pmod{\mathfrak{q}}$ be a Hecke character. For a polynomial $P(X) = \sum_{k=1}^d a_k X^k \in \mathbb{R}[X]$ of degree $d \geq 1$, define a real-valued function

$$(5.1) \quad \mathcal{P}(s, \chi) = \mathcal{P}(s, \chi; P) := \sum_{\mathfrak{n} \subseteq \mathcal{O}} \frac{\Lambda_K(\mathfrak{n})}{(\mathbf{N}\mathfrak{n})^\sigma} \left(\sum_{k=1}^d a_k \frac{((\sigma-1) \log \mathbf{N}\mathfrak{n})^{k-1}}{(k-1)!} \right) \cdot \operatorname{Re} \left\{ \frac{\chi(\mathfrak{n})}{(\mathbf{N}\mathfrak{n})^{it}} \right\}$$

for $\sigma > 1$ and where $\Lambda_K(\cdot)$ is the von Mangoldt Λ -function on integral ideals of \mathcal{O}_K defined by

$$\Lambda_K(\mathfrak{n}) = \begin{cases} \log \mathbf{N}\mathfrak{p} & \text{if } \mathfrak{n} \text{ is a power of a prime ideal } \mathfrak{p}, \\ 0 & \text{otherwise.} \end{cases}$$

From the classical formula

$$-\frac{L'}{L}(s, \chi) = \sum_{\mathfrak{n} \subseteq \mathcal{O}} \Lambda_K(\mathfrak{n}) \chi(\mathfrak{n}) (\mathbf{N}\mathfrak{n})^{-s} \quad \text{for } \sigma > 1,$$

it is straightforward to deduce that

$$(5.2) \quad \mathcal{P}(s, \chi) = \sum_{k=1}^d a_k (\sigma-1)^{k-1} \cdot \operatorname{Re} \left\{ \frac{(-1)^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \frac{L'}{L}(s, \chi) \right\} \quad \text{for } \sigma > 1.$$

To prove an explicit inequality using $\mathcal{P}(s, \chi)$, we first reduce the problem to primitive characters.

Lemma 5.1. *Let $\chi \pmod{\mathfrak{q}}$ be induced from the primitive character $\chi^* \pmod{\mathfrak{f}_\chi}$. Let $P(X) \in \mathbb{R}[X]$ be a polynomial with $P(0) = 0$. Then, for $\epsilon > 0$,*

$$\mathcal{P}(s, \chi) = \mathcal{P}(s, \chi^*) + O_P(\epsilon^{-1} n_K + \epsilon \mathcal{L})$$

uniformly in the region

$$1 < \sigma \leq 1 + \frac{100}{\mathcal{L}}.$$

Proof. Denote $d = \deg P$. Observe

$$\left| \mathcal{P}(s, \chi) - \mathcal{P}(s, \chi^*) \right| \ll_P \sum_{(\mathfrak{n}, \mathfrak{q}) \neq 1} \frac{\Lambda_K(\mathfrak{n})}{\mathbf{N}\mathfrak{n}} \left(\frac{\log \mathbf{N}\mathfrak{n}}{\mathcal{L}} \right)^{d-1} \ll_P \sum_{\mathfrak{p}|\mathfrak{q}} \sum_{j \geq 1} \frac{\log \mathbf{N}\mathfrak{p}}{(\mathbf{N}\mathfrak{p})^j} \cdot \left(\frac{j \log \mathbf{N}\mathfrak{p}}{\mathcal{L}} \right)^{d-1}.$$

For $\mathfrak{p} \mid \mathfrak{q}$, note $\log N\mathfrak{p} \ll \log N\mathfrak{q} \ll \mathcal{L}$ and so the above is

$$\ll_P \sum_{\mathfrak{p} \mid \mathfrak{q}} \sum_{j \geq 1} \frac{j^{d-1} \log N\mathfrak{p}}{(N\mathfrak{p})^j} \ll_P \sum_{\mathfrak{p} \mid \mathfrak{q}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}}.$$

The desired result then follows from Lemma 2.4. \square

Proposition 5.2. *Let $\chi \pmod{\mathfrak{q}}$ and $\epsilon > 0$ be arbitrary. Suppose the polynomial $P(X) = \sum_{k=1}^d a_k X^k$ of degree $d \geq 1$ has non-negative real coefficients. Then there exists $\delta = \delta(\epsilon, P) > 0$, such that*

$$(5.3) \quad \frac{1}{\mathcal{L}} \cdot \mathcal{P}(s, \chi) \leq \operatorname{Re} \left\{ \frac{P\left(\frac{\sigma-1}{s-1}\right)}{\sigma-1} E_0(\chi) - \sum_{|1+it-\rho_\chi| \leq \delta} \frac{P\left(\frac{\sigma-1}{s-\rho_\chi}\right)}{\sigma-1} \right\} \cdot \frac{1}{\mathcal{L}} + a_1 \phi \frac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon$$

uniformly in the region

$$1 + \frac{1}{\mathcal{L} \log \mathcal{L}} \leq \sigma \leq 1 + \frac{100}{\mathcal{L}}, \quad |t| \leq \mathcal{T}$$

provided \mathcal{L} is sufficiently large depending on ϵ and P .

Proof. Let $\chi^* \pmod{\mathfrak{f}_\chi}$ be the primitive character inducing $\chi \pmod{\mathfrak{q}}$. From Lemma 5.1 and the observation $n_K = o(\mathcal{L})$, it follows

$$\frac{1}{\mathcal{L}} \mathcal{P}(s, \chi) = \frac{1}{\mathcal{L}} \mathcal{P}(s, \chi^*) + \epsilon$$

for \mathcal{L} sufficiently large depending on ϵ and P . Thus, it suffices to show (5.3) with $\mathcal{P}(s, \chi^*)$ instead of $\mathcal{P}(s, \chi)$. Define

$$P_2(X) := \sum_{k=2}^d a_k X^k = P(X) - a_1 X.$$

Using Lemma 2.2 and Lemma 2.6, we see for $k \geq 2$ and $\sigma > 1$ that

$$\begin{aligned} \frac{(-1)^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \frac{L'}{L}(s, \chi^*) &= \frac{E_0(\chi)}{(s-1)^k} - \sum_{\rho_\chi} \frac{1}{(s-\rho_\chi)^k} + \frac{E_0(\chi)}{s^k} - \frac{(-1)^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \frac{L'_\infty}{L_\infty}(s, \chi^*) \\ &= \frac{E_0(\chi)}{(s-1)^k} - \sum_{\rho_\chi} \frac{1}{(s-\rho_\chi)^k} + O(n_K). \end{aligned}$$

Substituting these formulae into $\mathcal{P}(s, \chi^*; P_2)$ defined via (5.2), it follows for $\sigma > 1$ that

$$(5.4) \quad \mathcal{P}(s, \chi^*; P_2) = \frac{1}{\sigma-1} \sum_{k=2}^d a_k \operatorname{Re} \left\{ \left(\frac{\sigma-1}{s-1} \right)^k E_0(\chi) - \sum_{\rho_\chi} \left(\frac{\sigma-1}{s-\rho_\chi} \right)^k \right\} + O_P(n_K).$$

Obtain $\delta = \delta(\epsilon)$ from Lemma 4.3. Since $\sigma < 1 + \frac{100}{\mathcal{L}}$, we see by the zero density estimate [LMO79, Lemma 2.1] and Lemma 3.1 that

$$\begin{aligned} \sum_{|1+it-\rho_\chi| \geq \delta} \left| \frac{\sigma-1}{s-\rho_\chi} \right|^k &\ll \left(\frac{100}{\mathcal{L}} \right)^k \sum_{|1+it-\rho_\chi| \geq \delta} \frac{1}{|s-\rho_\chi|^k} \ll_\delta \left(\frac{100}{\mathcal{L}} \right)^k \sum_{\rho_\chi} \frac{1}{1+|t-\gamma_\chi|^2} \\ &\ll_\delta \left(\frac{100}{\mathcal{L}} \right)^k \cdot (\mathcal{L}^* + n_K \log \mathcal{T}) \\ &\ll_\delta \frac{(100)^k}{\mathcal{L}^{k-1}} \ll_{\delta, P} \frac{1}{\mathcal{L}}. \end{aligned}$$

Hence,

$$\frac{1}{\sigma-1} \sum_{k=2}^d a_k \operatorname{Re} \left\{ \sum_{|1+it-\rho_\chi| \geq \delta} \left(\frac{\sigma-1}{s-\rho_\chi} \right)^k \right\} \ll_{\epsilon, P} \log \mathcal{L}$$

since $\sigma > 1 + \frac{1}{\mathcal{L} \log \mathcal{L}}$ and δ depends only on ϵ . Removing this contribution in (5.4) implies that

$$\begin{aligned} \mathcal{P}(s, \chi^*; P_2) &= \frac{1}{\sigma-1} \sum_{k=2}^d a_k \operatorname{Re} \left\{ \left(\frac{\sigma-1}{s-1} \right)^k E_0(\chi) - \sum_{|1+it-\rho_\chi| \leq \delta} \left(\frac{\sigma-1}{s-\rho_\chi} \right)^k \right\} + O_{\epsilon, P}(n_K + \log \mathcal{L}) \\ &= \operatorname{Re} \left\{ \frac{P_2\left(\frac{\sigma-1}{s-1}\right)}{\sigma-1} E_0(\chi) - \sum_{|1+it-\rho_\chi| \leq \delta} \frac{P_2\left(\frac{\sigma-1}{s-\rho_\chi}\right)}{\sigma-1} \right\} + O_{\epsilon, P}(n_K + \log \mathcal{L}). \end{aligned}$$

For the linear polynomial $P_1(X) := a_1 X$, we apply Lemma 4.3 directly to find that

$$\mathcal{P}(s, \chi^*; P_1) \leq a_1(\phi + \epsilon) \mathcal{L}_\chi + \operatorname{Re} \left\{ \frac{P_1\left(\frac{\sigma-1}{s-1}\right)}{\sigma-1} E_0(\chi) - \sum_{|1+it-\rho_\chi| \leq \delta} \frac{P_1\left(\frac{\sigma-1}{s-\rho_\chi}\right)}{\sigma-1} \right\} + o_{\epsilon, P}(\mathcal{L})$$

for \mathcal{L} sufficiently large depending on ϵ .

Finally, from (5.2), we see that $\mathcal{P}(s, \chi^*; P) = \mathcal{P}(s, \chi^*; P_1) + \mathcal{P}(s, \chi^*; P_2)$ since $P = P_1 + P_2$, so combining the above inequality with the previous equation we conclude

$$\begin{aligned} \mathcal{P}(s, \chi^*) &\leq a_1(\phi + \epsilon) \mathcal{L}_\chi + \operatorname{Re} \left\{ \frac{P\left(\frac{\sigma-1}{s-1}\right)}{\sigma-1} E_0(\chi) - \sum_{|1+it-\rho_\chi| \leq \delta} \frac{P\left(\frac{\sigma-1}{s-\rho_\chi}\right)}{\sigma-1} \right\} \\ &\quad + O_{\epsilon, P}(n_K + \log \mathcal{L}) + o_{\epsilon, P}(\mathcal{L}). \end{aligned}$$

Dividing both sides by \mathcal{L} and taking \mathcal{L} sufficiently large depending on ϵ and P , the errors may be made arbitrarily small. Choosing a new ϵ yields the desired result. \square

We wish to use Proposition 5.2 in many contexts but typically we want to restrict the sum over zeros ρ to just a few specified zeros. To do so, we impose an additional condition on $P(X)$.

Definition 5.3. A polynomial $P(X) \in \mathbb{R}_{\geq 0}[X]$ is *admissible* if $P(0) = 0$ and

$$\operatorname{Re} \left\{ P\left(\frac{1}{z}\right) \right\} \geq 0 \quad \text{when } \operatorname{Re}\{z\} \geq 1.$$

Now we establish a general lemma which we will repeatedly apply in varying circumstances.

Lemma 5.4. *Let $\epsilon > 0$ and $0 < \lambda < 100$ be arbitrary, and let $s = \sigma + it$ with*

$$\sigma = 1 + \frac{\lambda}{\mathcal{L}}, \quad |t| \leq \mathcal{T}.$$

Let $\chi \pmod{\mathfrak{q}}$ be an arbitrary Hecke character and let $\mathcal{Z} := \{\tilde{\rho}_1, \tilde{\rho}_2, \dots, \tilde{\rho}_J\}$ be a finite multiset of zeros of $L(s, \chi)$ (called the extracted zeros) where

$$\tilde{\rho}_j = \tilde{\beta}_j + i\tilde{\gamma}_j = \left(1 - \frac{\tilde{\lambda}_j}{\mathcal{L}}\right) + i \cdot \frac{\tilde{\mu}_j}{\mathcal{L}}, \quad 1 \leq j \leq J.$$

Suppose $P(X) = \sum_{k=1}^d a_k X^k$ is an admissible polynomial. Then

$$\frac{\lambda}{\mathcal{L}} \cdot \mathcal{P}(s, \chi) \leq \operatorname{Re} \left\{ E_0(\chi) P\left(\frac{\lambda}{\lambda + i\mu}\right) - \sum_{j=1}^J P\left(\frac{\lambda}{\lambda + \tilde{\lambda}_j + i(\mu - \tilde{\mu}_j)}\right) \right\} + a_1 \lambda \phi \frac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon$$

for \mathcal{L} sufficiently large depending only on ϵ , λ , the polynomial P , and the number of extracted zeros J .

Proof. From Proposition 5.2 and the admissibility of p , it follows that
(5.5)

$$\frac{\lambda}{\mathcal{L}} \mathcal{P}(s, \chi) \leq a_1 \lambda \phi \frac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon + \operatorname{Re} \left\{ E_0(\chi) P\left(\frac{\lambda}{\lambda + i\mu}\right) - \sum_{\substack{|1+it-\rho_\chi| \leq \delta \\ \rho_\chi \in \mathcal{Z}}} P\left(\frac{\lambda}{\lambda + \lambda_\chi + i(\mu - \mu_\chi)}\right) \right\}$$

for some $\delta = \delta(\epsilon, p)$ and \mathcal{L} sufficiently large depending on ϵ , P and λ . Note the admissibility of P was used to restrict the sum over zeros further by throwing out $\rho_\chi \notin \mathcal{Z}$ satisfying $|1 + it - \rho_\chi| \leq \delta$. For the remaining sum, consider $\tilde{\rho}_j \in \mathcal{Z}$. If $|1 + it - \tilde{\rho}_j| \geq \delta$, then $|\tilde{\mu}_j - \mu| \gg_\delta \mathcal{L}$ or $\tilde{\lambda}_j \gg_\delta \mathcal{L}$. As $P(0) = 0$, it follows

$$\operatorname{Re} \left\{ P\left(\frac{\lambda}{\lambda + \tilde{\lambda}_j + i(\mu - \tilde{\mu}_j)}\right) \right\} \ll_{\epsilon, p, \lambda} \mathcal{L}^{-1}.$$

Hence, in the sum over zeros in (5.5), we may include each extracted zero $\tilde{\rho}_j$ with error $O_{\epsilon, P, \lambda}(\mathcal{L}^{-1})$ implying

$$\sum_{\substack{|1+it-\rho_\chi| \leq \delta \\ \rho_\chi \in \mathcal{Z}}} \operatorname{Re} \left\{ P\left(\frac{\lambda}{\lambda + \lambda_\chi + i(\mu - \mu_\chi)}\right) \right\} = \sum_{j=1}^J \operatorname{Re} \left\{ P\left(\frac{\lambda}{\lambda + \tilde{\lambda}_j + i(\mu - \tilde{\mu}_j)}\right) \right\} + O_{\epsilon, P, \lambda}(\mathcal{L}^{-1} J).$$

Using this estimate in (5.5) and taking \mathcal{L} sufficiently large depending on ϵ , λ , p and J , we have the desired result upon choosing a new ϵ . \square

During computations, we will employ Lemma 5.4 with $P(X) = P_4(X)$ as given in [HB95]. That is, for the remainder of this paper, denote

$$(5.6) \quad P_4(X) := X + X^2 + \frac{4}{5}X^3 + \frac{2}{5}X^4.$$

We establish a key property of $P_4(X)$ in Lemma 5.6 using the following observation.

Lemma 5.5. *Let $V, W \geq 0$ be arbitrary and $m \geq 1$ be a positive integer. Define*

$$G_m(x, y, z) := V \cdot \frac{x^m}{(x^2 + z^2)^m} + W \cdot \frac{y^m}{(y^2 + z^2)^m} - \frac{1}{(1 + z^2)^m}.$$

for $x, y, z \in \mathbb{R}$. If $x, y \geq 1$ then

$$G_m(x, y, z) \geq 0 \quad \text{provided} \quad \frac{V}{x^m} + \frac{W}{y^m} \geq 1.$$

Proof. Notice

$$G_m(x, y, z) = \frac{V/x^m}{(1 + (z/x)^2)^m} + \frac{W/y^m}{(1 + (z/y)^2)^m} - \frac{1}{(1 + z^2)^m} \geq \left(\frac{V}{x^m} + \frac{W}{y^m} - 1 \right) \frac{1}{(1 + z^2)^m}.$$

□

Lemma 5.6. *The polynomial $P_4(X)$ is admissible. Additionally, if $0 < a \leq b \leq c$, $A > 0$, and $B, C \geq 0$, then*

$$(5.7) \quad \operatorname{Re}\{C \cdot P_4\left(\frac{a}{c+it}\right) + B \cdot P_4\left(\frac{a}{b+it}\right) - A \cdot P_4\left(\frac{a}{a+it}\right)\} \geq 0$$

provided

$$\frac{C}{c^4} + \frac{B}{b^4} \geq \frac{A}{a^4}.$$

Proof. The proof that $P_4(X)$ is admissible is given in [HB95, Section 4]. It remains to prove (5.7). By direct computation, one can verify that

$$(5.8) \quad \operatorname{Re}\{P_4\left(\frac{a}{b+it}\right)\} = \frac{16}{5} \frac{(ab)^4}{(b^2 + t^2)^4} + \frac{a(b-a)}{5(b^2 + t^2)^3} Q(a, b, t),$$

where $Q(a, b, t) = 5t^4 + 2(5b^2 + 5ab - a^2)t^2 + b^2(5b^2 + 10ab + 14a^2)$ is clearly positive for $0 < a \leq b$ and $t \in \mathbb{R}$. Thus, for $0 < a \leq b$ and $t \in \mathbb{R}$, we have

$$(5.9) \quad \operatorname{Re}\{P_4\left(\frac{a}{b+it}\right)\} \geq \frac{16}{5} \frac{(ab)^4}{(b^2 + t^2)^4}.$$

Now, consider the LHS of (5.7). Apply (5.9) to the first and second term and (5.8) to the third term deducing that the LHS of (5.7) is

$$(5.10) \quad \geq \frac{16a^4}{5} \cdot \left(C \cdot \frac{c^4}{(c^2 + t^2)^4} + B \cdot \frac{b^4}{(b^2 + t^2)^4} - A \cdot \frac{a^4}{(a^2 + t^2)^4} \right) \geq \frac{16A}{5} \cdot G_4\left(\frac{c}{a}, \frac{b}{a}, \frac{t}{a}\right)$$

where $G_4(x, y, z)$ is defined in Lemma 5.5 with $V = C/A, W = B/A$. Applying Lemma 5.5 to $G_4\left(\frac{c}{a}, \frac{b}{a}, \frac{t}{a}\right)$ immediately implies (5.7) with the desired condition. □

6. SMOOTHED EXPLICIT INEQUALITY

We further generalize the “classical explicit inequality” to smoothly weighted versions of $-\frac{L'}{L}(s, \chi)$, similar to the well-known Weil’s explicit formula. For any Hecke character $\chi \pmod{\mathfrak{q}}$ and function $f : [0, \infty) \rightarrow \mathbb{R}$ with compact support, define

$$\mathcal{W}(s, \chi; f) := \sum_{\mathfrak{n} \subseteq \mathcal{O}} \Lambda_K(\mathfrak{n}) \chi(\mathfrak{n}) (\mathbf{N}\mathfrak{n})^{-s} f\left(\frac{\log(\mathbf{N}\mathfrak{n})}{\mathcal{L}}\right) \quad \text{for } \sigma > 1,$$

$$\mathcal{K}(s, \chi; f) := \operatorname{Re}\{\mathcal{W}(s, \chi; f)\}.$$

We begin with the same setup as [HB95, Section 5]. Assume f satisfies the following condition:

Condition 1 Let f be a continuous function from $[0, \infty)$ to \mathbb{R} , supported in $[0, x_0)$ and bounded absolutely by M , and let f be twice differentiable on $(0, x_0)$, with f'' being continuous and bounded by B .

Recall that the Laplace transform of f is given by

$$(6.1) \quad F(z) := \int_0^\infty e^{-zt} f(t) dt, \quad z \in \mathbb{C}.$$

Note $F(z)$ is entire since f has compact support. For $\operatorname{Re}(z) > 0$, we have

$$(6.2) \quad F(z) = \frac{1}{z} f(0) + F_0(z),$$

where

$$(6.3) \quad |F_0(z)| \leq |z|^{-2} A(f)$$

with

$$A(f) = 3Bx_0 + \frac{2|f(0)|}{x_0}.$$

Define the content of f to be

$$(6.4) \quad \mathcal{C} = \mathcal{C}(f) := (x_0, M, B, f(0)).$$

For the purposes of generality, estimates in this section will depend only on the content of f . For all subsequent sections, we will ignore this distinction and allow dependence on f in general. We first reduce our analysis to primitive characters and then prove the main result.

Lemma 6.1. Suppose $\chi \pmod{\mathfrak{q}}$ is induced from $\chi^* \pmod{\mathfrak{f}_\chi}$. For $\epsilon > 0$ and f satisfying Condition 1,

$$\mathcal{W}(s, \chi; f) = \mathcal{W}(s, \chi^*; f) + O_c\left(\frac{n_K}{\sqrt{\epsilon}} + \epsilon \mathcal{L}^*\right)$$

uniformly in the region $\sigma > 1$.

Proof. Observe

$$\begin{aligned} \left| \mathcal{W}(s, \chi; f) - \mathcal{W}(s, \chi^*; f) \right| &\leq \sum_{(\mathfrak{n}, \mathfrak{q}) \neq 1} \frac{\Lambda_K(\mathfrak{n})}{N\mathfrak{n}} |f(\mathcal{L}^{-1}(\log N\mathfrak{n}))| \\ &\leq M \sum_{(\mathfrak{n}, \mathfrak{q}) \neq 1} \frac{\Lambda_K(\mathfrak{n})}{N\mathfrak{n}} = M \sum_{\mathfrak{p}|\mathfrak{q}} \sum_{j \geq 1} \frac{\log N\mathfrak{p}}{(N\mathfrak{p})^j} \leq 2M \sum_{\mathfrak{p}|\mathfrak{q}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}}. \end{aligned}$$

The desired result then follows from Lemma 2.4. \square

Proposition 6.2. Let $\chi \pmod{\mathfrak{q}}$ and $\epsilon > 0$ be arbitrary, and suppose $s = \sigma + it$ satisfies

$$|\sigma - 1| \leq \frac{(\log \mathcal{L})^{1/2}}{\mathcal{L}}, \quad |t| \leq \mathcal{T}.$$

Suppose f satisfies Condition 1 and that $f(0) \geq 0$. Then there exists $\delta = \delta(\mathcal{C}, \epsilon) \in (0, 1)$ depending only on ϵ and the content of f (and independent of χ, \mathfrak{q}, K and s) such that

$$(6.5) \quad \begin{aligned} \frac{1}{\mathcal{L}} \cdot \mathcal{K}(s, \chi; f) &\leq E_0(\chi) \cdot \operatorname{Re}\{F((s-1)\mathcal{L})\} - \sum_{|1+it-\rho| \leq \delta} \operatorname{Re}\{F((s-\rho)\mathcal{L})\} \\ &\quad + f(0)\phi \frac{\mathcal{L}_x}{\mathcal{L}} + \epsilon \end{aligned}$$

provided \mathcal{L} is sufficiently large depending on ϵ and the content of f .

Proof. The proof will closely follow the arguments of [HB95, Lemma 5.2]. Let $\chi^* \pmod{\mathfrak{f}_\chi}$ be the primitive character inducing χ . From Lemma 6.1,

$$\mathcal{K}(s, \chi; f) = \mathcal{K}(s, \chi^*; f) + O_C(\epsilon^{-1}n_K + \epsilon\mathcal{L}^*).$$

Dividing both sides by \mathcal{L} and recalling $n_K = o(\mathcal{L})$, it follows that

$$\mathcal{L}^{-1}\mathcal{K}(s, \chi; f) \leq \mathcal{L}^{-1}\mathcal{K}(s, \chi^*; f) + \epsilon$$

for \mathcal{L} sufficiently large depending on ϵ and the content of f . Thus, we may prove (6.5) with $\mathcal{K}(s, \chi^*; f)$ instead of $\mathcal{K}(s, \chi; f)$.

Let $\sigma \geq 1 + 2\mathcal{L}^{-1}$ and set $\sigma_0 := 1 + \mathcal{L}^{-1}$ so $\sigma_0 < \sigma$. Consider

$$(6.6) \quad I := \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left(-\frac{L'}{L}(w, \chi^*) \right) F_0((s-w)\mathcal{L}) dw.$$

Since F_0 satisfies (6.3) and

$$-\frac{L'}{L}(w, \chi^*) \ll \left| \frac{\zeta'_K}{\zeta_K}(\sigma_0) \right| \ll n_K(\sigma_0 - 1)^{-1}$$

by Lemma 2.5, the integral converges absolutely. Hence, we may compute I by interchanging the summation and integration, and calculating the integral against $(N\mathfrak{n})^{-w}$ term-wise. That is to say,

$$(6.7) \quad I = \sum_{\mathfrak{n} \subseteq \mathcal{O}} \Lambda(\mathfrak{n}) \chi^*(\mathfrak{n}) \left(\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} (N\mathfrak{n})^{-w} F_0((s-w)\mathcal{L}) dw \right).$$

Arguing as in [HB95, Section 5, p.21] and using Lebesgue's Dominated Convergence Theorem, one can verify

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} (N\mathfrak{n})^{-w} F_0((s-w)\mathcal{L}) dw = \frac{(N\mathfrak{n})^{-s}}{\mathcal{L}} \cdot (f(\mathcal{L}^{-1} \log N\mathfrak{n}) - f(0))$$

since f satisfies Condition 1. Substituting this result into (6.7), we see that

$$(6.8) \quad I = \frac{1}{\mathcal{L}} \left(\mathcal{W}(s, \chi^*; f) + \frac{L'}{L}(s, \chi^*) f(0) \right).$$

Returning to (6.6), we shift the line of integration from $(\sigma_0 \pm \infty)$ to $(-\frac{1}{2} \pm \infty)$ yielding

$$(6.9) \quad \begin{aligned} I &= E_0(\chi) F_0((s-1)\mathcal{L}) - \sum_{\rho} F_0((s-\rho)\mathcal{L}) \\ &\quad - r(\chi) F_0(s\mathcal{L}) + \frac{1}{2\pi i} \int_{-\frac{1}{2} - i\infty}^{-\frac{1}{2} + i\infty} \left(-\frac{L'}{L}(w, \chi^*) \right) F_0((s-w)\mathcal{L}) dw \end{aligned}$$

where the sum is over the non-trivial zeros of $L(w, \chi)$ and $r(\chi) \geq 0$ is the order of the trivial zero $w = 0$ of $L(w, \chi^*)$. From (2.1), notice $r(\chi) \leq n_K$ so by (6.3),

$$r(\chi) |F_0(s\mathcal{L})| \ll \frac{n_K A(f)}{|s\mathcal{L}|^2} \ll \frac{n_K A(f)}{\mathcal{L}^2} \ll \frac{A(f)}{\mathcal{L}}.$$

To bound the remaining integral in (6.9), we apply the functional equation (2.3) of $L(w, \chi^*)$ and Lemma 2.3; namely, we note for $\operatorname{Re}\{w\} = -1/2$ that

$$-\frac{L'}{L}(w, \chi^*) = \mathcal{L}_\chi^* + \frac{L'}{L}(1-w, \overline{\chi^*}) + O(n_K \log(2+|w|)) = \mathcal{L}_\chi^* + O(n_K \log(2+|w|))$$

using Lemma 2.5 since $\operatorname{Re}\{1-w\} = 3/2$. From (6.3), we therefore find that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \left(-\frac{L'}{L}(w, \chi^*) \right) F_0((s-w)\mathcal{L}) dw \\ &= \frac{\mathcal{L}_\chi^*}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} F_0((s-w)\mathcal{L}) dw + O\left(\frac{A(f)}{\mathcal{L}^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{n_K \cdot \log(2+|w|)}{|s-w|^2} dw \right) \end{aligned}$$

Since F_0 is entire and satisfies (6.3), we may pull the line of integration in the first integral as far left as we desire, concluding that the first integral vanishes. One can readily verify that integral in the error term is

$$\ll \frac{A(f)}{\mathcal{L}^2} \cdot n_K \log(2+|s|) \ll \frac{A(f)n_K \log \mathcal{T}}{\mathcal{L}^2} \ll \frac{A(f)}{\mathcal{L}}$$

by Lemma 3.1. Combining these bounds into (6.9) and comparing with (6.8), we deduce (6.10)

$$\frac{1}{\mathcal{L}} \cdot \mathcal{W}(s, \chi^*; f) = -\frac{L'}{L}(s, \chi^*) f(0) \frac{1}{\mathcal{L}} + E_0(\chi) \cdot F_0((s-1)\mathcal{L}) - \sum_{\rho} F_0((s-\rho)\mathcal{L}) + O\left(\frac{A(f)}{\mathcal{L}}\right).$$

We wish to apply Lemma 4.3 giving $\delta = \delta(\epsilon)$, but we must discard zeros in the above sum where $|1+it-\rho| \geq \delta$. By (6.3), [LMO79, Lemma 2.1], and Lemma 3.1, this discard induces an error

$$\ll \sum_{|1+it-\rho| \geq \delta} \frac{A(f)}{\mathcal{L}^2 |s-\rho|^2} \ll_{\delta} \frac{A(f)}{\mathcal{L}^2} \sum_{\rho} \frac{1}{1+|t-\gamma|^2} \ll_{\delta} \frac{A(f)}{\mathcal{L}^2} (\mathcal{L}^* + n_K \log \mathcal{T}) \ll_{\delta} \frac{A(f)}{\mathcal{L}}.$$

Hence, taking real parts of (6.10), applying Lemma 4.3, and using (6.2), we find

$$\frac{\mathcal{K}(s, \chi^*; f)}{\mathcal{L}} \leq E_0(\chi) \operatorname{Re} \left\{ F((s-1)\mathcal{L}) - \sum_{|1+it-\rho| < \delta} F((s-\rho)\mathcal{L}) \right\} + f(0) (\phi + \epsilon) \frac{\mathcal{L}_\chi}{\mathcal{L}} + O_{\delta, \mathcal{L}}(\mathcal{L}^{-1}).$$

Taking \mathcal{L} sufficiently large depending on ϵ and the content of f , the error term may be made arbitrarily small. Upon choosing a new ϵ , we have established (6.5) in the range

$$1 + 2\mathcal{L}^{-1} \leq \sigma \leq 1 + (\log \mathcal{L})^{1/2} \mathcal{L}^{-1}.$$

Similar to the discussion in [HB95, Section 5, p.22-23], one may show (6.5) holds in the desired extended range by considering $g(t) = e^{\alpha t} f(t)$ for $0 \leq \alpha \leq (\log \mathcal{L})/3x_0$. \square

In analogy with Proposition 5.2 and Lemma 5.4, we would like to use Proposition 6.2 restricting the sum over zeros ρ to just a few specified zeros. To do so, we require our weight f to satisfy an additional condition which was introduced in [HB95, Section 6].

Condition 2 *The function f is non-negative. Moreover, its Laplace transform F satisfies*

$$\operatorname{Re}\{F(z)\} \geq 0 \text{ for } \operatorname{Re}(z) \geq 0.$$

Condition 2 implies that, viewed as a real-variable function of $t \in \mathbb{R}$, $F(t)$ is a positive decreasing real-valued function. We may now give a more convenient version of Proposition 6.2 in the following lemma.

Lemma 6.3. *Let $\epsilon \in (0, 1)$ be arbitrary, and let $s = \sigma + it$ with*

$$|\sigma - 1| \leq \frac{\log \log \mathcal{T}}{C_0 \mathcal{L}}, \quad |t| \leq 5T_0$$

where constants $C_0 > 0$ and $T_0 \geq 1$ come from Lemma 3.2. Write $\sigma = 1 - \lambda/\mathcal{L}$ and $t = \mu/\mathcal{L}$.

Let $\chi \pmod{\mathfrak{q}}$ be an arbitrary Hecke character and let $\mathcal{Z} := \{\tilde{\rho}_1, \tilde{\rho}_2, \dots, \tilde{\rho}_J\}$ be a finite, possibly empty, multiset of zeros of $L(s, \chi)$ (called the extracted zeros) containing the multiset

$$\{\rho_\chi : \sigma < \beta_\chi \leq 1, \quad |\gamma_\chi| \leq T_0\}.$$

Write $\tilde{\rho}_j = \tilde{\beta}_j + i\tilde{\gamma}_j = \left(1 - \frac{\tilde{\lambda}_j}{\mathcal{L}}\right) + i \cdot \frac{\tilde{\mu}_j}{\mathcal{L}}$ for $1 \leq j \leq J$ and suppose f satisfies Conditions 1 and 2. Then

$$\mathcal{L}^{-1} \cdot \mathcal{K}(s, \chi; f) \leq E_0(\chi) \cdot \operatorname{Re}\{F(-\lambda + i\mu)\} - \sum_{j=1}^J \operatorname{Re}\{F(\tilde{\lambda}_j - \lambda - i(\tilde{\mu}_j - \mu))\} + f(0)\phi\frac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon$$

for \mathcal{L} sufficiently large depending only on ϵ , the content of f , and the number of extracted zeros J .

Remark. The dependence of “sufficiently large” on J is insignificant for our purposes, as we will employ the lemma with $0 \leq J \leq 10$ in all of our applications.

Proof. From Proposition 6.2, it follows that

$$(6.11) \quad \frac{\mathcal{K}(s, \chi; f)}{\mathcal{L}} \leq f(0)\left(\phi\frac{\mathcal{L}_\chi}{\mathcal{L}} + \epsilon\right) + E_0(\chi) \cdot \operatorname{Re}\{F(-\lambda + i\mu)\} - \sum_{|1+it-\rho| \leq \delta} \operatorname{Re}\{F((s-\rho)\mathcal{L})\}$$

for some $\delta = \delta(\epsilon, \mathcal{C})$. We consider the sum over zeros depending on whether $\rho \in \mathcal{Z}$ or not. For any $\rho = \tilde{\rho}_j \in \mathcal{Z}$, if $|1 + it - \tilde{\rho}_j| \geq \delta$, then $|\tilde{\mu}_j - \mu| \gg_\delta \mathcal{L}$ or $\tilde{\lambda}_j \gg_\delta \mathcal{L}$. From (6.2) and (6.3), it follows that

$$\operatorname{Re}\{F((s - \tilde{\rho}_j)\mathcal{L})\} \ll_{\epsilon, \mathcal{C}} \mathcal{L}^{-1}.$$

implying

$$(6.12) \quad \sum_{\substack{|1+it-\rho| \leq \delta \\ \rho \in \mathcal{Z}}} \operatorname{Re}\{F((s-\rho)\mathcal{L})\} = \sum_{j=1}^J \operatorname{Re}\{F(\tilde{\lambda}_j - \lambda - i(\tilde{\mu}_j - \mu))\} + O_{\epsilon, \mathcal{C}}(J\mathcal{L}^{-1}).$$

Next, for all zeros $\rho = \beta + i\gamma \notin \mathcal{Z}$ satisfying $|1 + it - \rho| \leq \delta$, we claim $\beta \leq \sigma$. Assuming the claim, it follows by Condition 2 that

$$(6.13) \quad \sum_{\substack{|1+it-\rho| \leq \delta \\ \rho \notin \mathcal{Z}}} \operatorname{Re}\{F((s-\rho)\mathcal{L})\} \geq 0.$$

To see the claim, assume for a contradiction that $\sigma \leq \beta \leq 1$ for some zero $\rho = \beta + i\gamma$ occurring in (6.13). As $|1 + it - \rho| \leq \delta$, it follows that

$$|\gamma| \leq |t| + \delta \leq 5T_0 + 1 \leq 6T_0.$$

From Lemma 3.2, either

$$|\gamma| \leq T_0 \quad \text{or} \quad \beta \leq 1 - \frac{\log \log \mathcal{T}}{C_0 \mathcal{L}}.$$

In the latter case, it follows $\beta \leq \sigma$ which is a contradiction, so it must be that $|\gamma| \leq T_0$ and $\sigma \leq \beta \leq 1$. By the assumptions of the lemma, it follows $\rho \in \mathcal{Z}$, which is also a contradiction. This proves the claim.

Therefore, combining (6.12) and (6.13), we may conclude

$$- \sum_{|1+it-\rho| \leq \delta} \operatorname{Re}\{F((s-\rho)\mathcal{L})\} \leq - \sum_{j=1}^J \operatorname{Re}\{F(\tilde{\lambda}_j - \lambda - i(\tilde{\mu}_j - \mu))\} + O_{\epsilon, \mathcal{C}}(J\mathcal{L}^{-1}).$$

Using this bound in (6.11) and taking \mathcal{L} sufficiently large depending on ϵ, \mathcal{C} and J , we have the desired result upon choosing a new ϵ . \square

We also record a lemma useful for applications of Lemma 6.3 in Sections 8 and 9.

Lemma 6.4. *Suppose f satisfies Conditions 1 and 2. For $a, b \geq 0$ and $y \in \mathbb{R}$, we have that*

$$\operatorname{Re}\{F(-a + iy) - F(iy) - F(b - a + iy)\} \leq \begin{cases} F(-a) - F(0) & \text{if } b \geq a, \\ F(-a) - F(b - a) & \text{if } b \leq a. \end{cases}$$

Proof. If $b \geq a$, then by Condition 2, $\operatorname{Re}\{F(b - a + iy)\} \geq 0$ so the LHS of the desired inequality is

$$\leq \operatorname{Re}\{F(-a + iy) - F(iy)\} = \int_0^\infty f(t)(e^{at} - 1) \cos(yt) dt \leq \int_0^\infty f(t)(e^{at} - 1) dt = F(-a) - F(0)$$

since $f(t) \geq 0$ and $a \geq 0$. A similar argument holds for $b \leq a$, except we exclude $\operatorname{Re}\{F(iy)\}$ in this case. \square

7. NUMERICAL ZERO DENSITY ESTIMATE

Let us first introduce some notation intended only for this section.

Worst Low-lying Zeros of each Character

Consider the rectangle

$$\{s \in \mathbb{C} : 0 \leq \sigma \leq 1, \quad |t| \leq 1\}.$$

For each character with a zero in this rectangle, index it $\chi^{(k)}$ for $k = 1, 2, \dots$ with a zero $\rho^{(k)}$ in this rectangle defined by:

$$\operatorname{Re}(\rho^{(k)}) = \max\{\operatorname{Re}(\rho) : L(\rho, \chi^{(k)}) = 0, |\gamma| \leq 1\},$$

so $\chi^{(j)} \neq \chi^{(k)}$ for $j \neq k$. Write

$$\rho^{(k)} := \beta^{(k)} + i\gamma^{(k)}, \quad \beta^{(k)} = 1 - \frac{\lambda^{(k)}}{\mathcal{L}}, \quad \gamma^{(k)} = \frac{\mu^{(k)}}{\mathcal{L}}.$$

Without loss, we may assume $\lambda^{(1)} \leq \lambda^{(2)} \leq \dots$ and so on.

Remark. Upon comparing with the indexing given in Section 3, we always have the bound $\lambda_k \leq \lambda^{(k)}$ for all k where both quantities exist.

Low-lying Zero Density

For $\lambda \geq 0$, consider the rectangle

$$\mathcal{S} = \mathcal{S}(\lambda) := \{s \in \mathbb{C} : 1 - \frac{\lambda}{\mathcal{L}} \leq \sigma \leq 1, \quad |t| \leq 1\}.$$

Define

$$N = N(\lambda) := \#\{\chi \neq \chi_0 \pmod{\mathfrak{q}} \mid L(s, \chi) \text{ has a zero in } \mathcal{S}(\lambda)\} = \sum_{\substack{\lambda^{(k)} \leq \lambda \\ \chi^{(k)} \neq \chi_0}} 1$$

Below is the main result of this section which gives bounds on $N(\lambda)$ using the smoothed explicit inequality.

Theorem 7.1. *Suppose f satisfies Conditions 1 and 2 and let $\epsilon > 0$. Assume $\lambda_1 \geq b$ for some $b \geq 0$. For $\lambda \geq 0$, if*

$$F(\lambda - b) > \frac{1}{\vartheta} f(0)\phi,$$

and

$$\left(F(\lambda - b) - \frac{1}{\vartheta} f(0)\phi\right)^2 > \frac{1}{\vartheta} f(0)\phi \left(f(0)\phi + F(-b)\right)$$

then unconditionally,

$$(7.1) \quad N(\lambda) \leq \frac{\left(f(0)\phi + F(-b)\right) \left(F(-b) - \left(\frac{1}{\vartheta} - 1\right)f(0)\phi\right)}{\left(F(\lambda - b) - \frac{1}{\vartheta} f(0)\phi\right)^2 - \frac{1}{\vartheta} f(0)\phi \left(f(0)\phi + F(-b)\right)} + \epsilon$$

for \mathcal{L} sufficiently large depending on ϵ and f .

Remark. Recall $\vartheta \in [\frac{3}{4}, 1]$ by the definition of \mathcal{L} in Section 3.

Remark. If $\zeta_K(s)$ has a real zero in $\mathcal{S}(\lambda)$, then one can extract this zero from $\mathcal{K}(\sigma, \chi_0; f)$ in the argument below and hence improve (7.1) to

$$N(\lambda) \leq \frac{\left(f(0)\phi + F(-b) - F(\lambda - b)\right) \left(F(-b) - F(\lambda - b) - \left(\frac{1}{\vartheta} - 1\right)f(0)\phi\right)}{\left(F(\lambda - b) - \frac{1}{\vartheta} f(0)\phi\right)^2 - \frac{1}{\vartheta} f(0)\phi \left(f(0)\phi + F(-b) - F(\lambda - b)\right)} + \epsilon$$

with naturally modified assumptions. The utility of such a bound is not entirely clear. If the real zero is exceptional, then the Deuring-Heilbronn phenomenon from Section 8 would likely be a better substitute.

Proof. We closely follow the arguments in [HB95, Section 12]. Let $\chi \pmod{\mathfrak{q}}$ denote a non-principal character with a zero $\tilde{\rho} = \tilde{\beta} + i\tilde{\gamma}$ in $\mathcal{S}(\lambda)$; that is, $b \leq \lambda_1 \leq \tilde{\lambda} \leq \lambda$. Applying Lemma 6.3 with $s = \sigma + i\tilde{\gamma}$ where $\sigma = 1 - \frac{b}{\mathcal{L}}$ and $\mathcal{Z} = \{\tilde{\rho}\}$ we find that

$$(7.2) \quad \mathcal{L}^{-1} \cdot \mathcal{K}(\sigma + i\tilde{\gamma}, \chi; f) \leq f(0)\phi \frac{\mathcal{L}_\chi}{\mathcal{L}} - F(\tilde{\lambda} - b) + \epsilon$$

for \mathcal{L} sufficiently large depending on ϵ and the content of f . Since F is decreasing by Condition 2, it follows that $F(\tilde{\lambda} - b) \geq F(\lambda - b)$. Also recalling that $\frac{\mathcal{L}_x}{\mathcal{L}} \leq \vartheta^{-1}$ by (3.2) and (3.3), we see that (7.2) implies:

$$(7.3) \quad \mathcal{L}^{-1} \cdot \mathcal{K}(\sigma + i\tilde{\gamma}, \chi; f) \leq f(0)\frac{1}{\vartheta}\phi - F(\lambda - b) + \epsilon.$$

Summing (7.3) over $\chi = \chi^{(j)}$ (which are non-principal by construction) and $\tilde{\gamma} = \gamma^{(j)}$ for $j = 1, \dots, N$ where $N = N(\lambda)$, we deduce that

$$(7.4) \quad \begin{aligned} \left(F(\lambda - b) - f(0)\frac{1}{\vartheta}\phi - \epsilon\right)N\mathcal{L} &\leq - \sum_{j \leq N} \mathcal{K}(\sigma + i\gamma^{(j)}, \chi^{(j)}; f) \\ &= - \sum_{(\mathbf{n}, \mathbf{q})=1} \Lambda(\mathbf{n})(N\mathbf{n})^{-\sigma} f\left(\frac{\log N\mathbf{n}}{\mathcal{L}}\right) \operatorname{Re}\left\{ \sum_{j \leq N} \chi^{(j)}(\mathbf{n})(N\mathbf{n})^{-i\gamma^{(j)}} \right\} \\ &\leq \sum_{(\mathbf{n}, \mathbf{q})=1} \Lambda(\mathbf{n})(N\mathbf{n})^{-\sigma} f\left(\frac{\log N\mathbf{n}}{\mathcal{L}}\right) \left| \sum_{j \leq N} \chi^{(j)}(\mathbf{n})(N\mathbf{n})^{-i\gamma^{(j)}} \right|. \end{aligned}$$

The LHS of (7.4) is positive by assumption so after squaring both sides of (7.4), we apply Cauchy-Schwarz to the last expression on the RHS implying

$$(\text{LHS of (7.4)})^2 \leq S_1 S_2$$

where

$$\begin{aligned} S_1 &= \sum_{(\mathbf{n}, \mathbf{q})=1} \Lambda(\mathbf{n})(N\mathbf{n})^{-\sigma} f\left(\frac{\log N\mathbf{n}}{\mathcal{L}}\right) = \mathcal{K}(\beta, \chi_0; f), \\ \text{and} \quad S_2 &= \sum_{(\mathbf{n}, \mathbf{q})=1} \Lambda(\mathbf{n})(N\mathbf{n})^{-\sigma} f\left(\frac{\log N\mathbf{n}}{\mathcal{L}}\right) \left| \sum_{j \leq N} \chi^{(j)}(\mathbf{n})(N\mathbf{n})^{-i\gamma^{(j)}} \right|^2 \\ &= \sum_{j, k \leq N} \mathcal{K}(\sigma + i(\gamma^{(j)} - \gamma^{(k)}), \chi^{(j)}\overline{\chi}^{(k)}; f). \end{aligned}$$

The 1 term from S_1 and the N terms in S_2 with $j = k$ give

$$\mathcal{K}(\sigma, \chi_0; f) \leq \mathcal{L}(f(0)\phi + F(-b) + \epsilon)$$

by Lemma 6.3. For the $N^2 - N$ terms in S_2 with $j \neq k$, apply Lemma 6.3 extracting no zeros to see that

$$\mathcal{K}(\sigma + i(\gamma^{(j)} - \gamma^{(k)}), \chi^{(j)}\overline{\chi}^{(k)}; f) \leq \mathcal{L}(f(0)\frac{1}{\vartheta}\phi + \epsilon).$$

Therefore, from (7.4), we conclude

$$\begin{aligned} &\left(F(\lambda - b) - f(0)\frac{1}{\vartheta}\phi - \epsilon\right)^2 N^2 \mathcal{L}^2 \\ &\leq \mathcal{L}\left[f(0)\phi + \epsilon + F(-b)\right] \times \mathcal{L}\left[\left(f(0)\phi + \epsilon + F(-b)\right)N + \left(f(0)\frac{1}{\vartheta}\phi + \epsilon\right)(N^2 - N)\right] \end{aligned}$$

Dividing both sides by $N\mathcal{L}^2$, solving the inequality, and choosing a new $\epsilon > 0$ depending on f , we find

$$N \leq \frac{\left(f(0)\phi + F(-b)\right)\left(F(-b) - \left(\frac{1}{\vartheta} - 1\right)f(0)\phi\right)}{\left(F(\lambda - b) - \frac{1}{\vartheta}f(0)\phi\right)^2 - \frac{1}{\vartheta}f(0)\phi\left(f(0)\phi + F(-b)\right)} + \epsilon$$

	$\lambda_1 \geq 0$	$\lambda_1 \geq .0875$	$\lambda_1 \geq .1$	$\lambda_1 \geq .1227$	$\lambda_1 \geq .15$	$\lambda_1 \geq .20$	$\lambda_1 \geq .25$	$\lambda_1 \geq .30$	$\lambda_1 \geq .35$
λ	$N(\lambda)$	$N(\lambda)$	$N(\lambda)$	$N(\lambda)$	$N(\lambda)$	$N(\lambda)$	$N(\lambda)$	$N(\lambda)$	$N(\lambda)$
.1	2	2							
.125	2	2	2	2					
.150	3	3	3	3					
.175	3	3	3	3	3				
.200	4	4	4	3	3				
.225	4	4	4	4	4	4			
.250	5	5	5	5	4	4			
.275	6	6	5	5	5	5	5		
.300	7	6	6	6	6	6	5		
.325	9	8	7	7	7	7	6	6	
.350	11	9	9	9	8	8	7	7	
.375	15	11	11	10	10	9	8	8	7
.400	22	15	14	13	12	11	10	9	8
.425	46	22	20	18	16	14	12	11	10
.450	∞	41	36	29	24	19	16	13	12
.475		1087	207	85	51	30	22	18	15
.500		∞	∞	∞	∞	90	40	27	21
.525						∞	413	61	34
.550							∞	∞	127
.575									∞
.600									

TABLE 1. Bounds for $N(\lambda)$

provided the denominator is > 0 which is one of our hypotheses. \square

To demonstrate the utility of Theorem 7.1, we produce a table of numerical bounds for $N(\lambda)$. Just as in Heath-Brown's case [HB95, Table 13], it turns out that the acquired bounds only hold for certain bounded ranges of $\lambda \in [0, \lambda_b]$ depending on $\lambda_1 \geq b$. However, for small values of λ , the resulting bounds are expected to be better than an explicit version of classical zero density estimates which has yet to be established.

We apply Theorem 7.1 using $\vartheta = \frac{3}{4}$. From the definitions of \mathcal{L} , $N(\lambda)$ and $\mathcal{S}(\lambda)$, it is immediate that the same bounds hold for all $\vartheta \in [\frac{3}{4}, 1]$. Choose the weight $f = f_{\hat{\theta}, \hat{\lambda}}$ from [HB95, Lemma 7.1] with parameters $\hat{\theta}$ and $\hat{\lambda}$, say, taking

$$\hat{\theta} = 1.63 + 1.28b - 4.35\lambda, \quad \hat{\lambda} = \lambda.$$

This is roughly optimal based on numerical experimentation and produces Table 1. Only non-trivial bounds are displayed since trivially $N(\lambda) \leq 1$ for $\lambda < \lambda_1$.

8. ZERO REPULSION: χ_1 AND ρ_1 ARE REAL

Recall the indexing of zeros from Section 3. Throughout this section, we assume χ_1 and ρ_1 are real. We wish to quantify the zero repulsion (also called Deuring-Heilbronn phenomenon) of ρ_1 with ρ' and ρ_2 using the results of Sections 5 and 6 along with various trigonometric

identities analogous to the classical one: $3 + 4 \cos \theta + \cos 2\theta \geq 0$. We emphasize that χ_1 can be quadratic or possibly principal.

We will primarily use the smoothed explicit inequality (Lemma 6.3) and so we assume that the weight function f continues to satisfy Conditions 1 and 2. For simplicity, henceforth denote $\mathcal{K}(s, \chi) = \mathcal{K}(s, \chi; f)$. Suppose characters χ, χ_* have zeros ρ, ρ_* respectively. Our starting point is the trigonometric identity

$$0 \leq \chi_0(\mathbf{n}) \left(1 + \operatorname{Re}\{\chi(\mathbf{n})(\mathbf{Nn})^{i\gamma}\}\right) \left(1 + \operatorname{Re}\{\chi_*(\mathbf{n})(\mathbf{Nn})^{i\gamma_*}\}\right).$$

Multiplying by $\Lambda(\mathbf{n})f(\mathcal{L}^{-1} \log \mathbf{Nn})(\mathbf{Nn})^{-\sigma}$ and summing over \mathbf{n} , it follows that

$$(8.1) \quad \begin{aligned} 0 \leq & \mathcal{K}(\sigma, \chi_0) + \mathcal{K}(\sigma + i\gamma, \chi) + \mathcal{K}(\sigma + i\gamma_*, \chi_*) \\ & + \frac{1}{2}\mathcal{K}(\sigma + i\gamma + i\gamma_*, \chi\chi_*) + \frac{1}{2}\mathcal{K}(\sigma + i\gamma - i\gamma_*, \chi\overline{\chi_*}) \end{aligned} \quad \text{for } \sigma > 0.$$

In some cases, we will use a simpler trigonometric identity:

$$0 \leq \chi_0(\mathbf{n}) + \operatorname{Re}\{\chi(\mathbf{n})(\mathbf{Nn})^{i\gamma}\}$$

which similarly yields

$$(8.2) \quad 0 \leq \mathcal{K}(\sigma, \chi_0) + \mathcal{K}(\sigma + i\gamma, \chi) \quad \text{for } \sigma > 0.$$

8.1. Bounds for λ' . We establish zero repulsion results for ρ' in terms of ρ_1 , using different methods depending on various ranges of λ_1 . In this subsection, we intentionally include more details to proofs but in later subsections we shall omit these extra explanations as the arguments will be similar to those found here.

Lemma 8.1. *Assume χ_1 and ρ_1 are real. Let $\epsilon > 0$ and suppose f satisfies Conditions 1 and 2. Provided \mathcal{L} is sufficiently large depending on ϵ and f , the following holds:*

(a) *If χ_1 is quadratic and $\lambda' \leq \lambda_2$, then with $\psi = 4\phi$ it follows that*

$$0 \leq F(-\lambda') - F(0) - F(\lambda_1 - \lambda') + \operatorname{Re}\{F(-\lambda' + i\mu') - F(i\mu') - F(\lambda_1 - \lambda' + i\mu')\} + f(0)\psi + \epsilon.$$

(b) *If χ_1 is principal, then with $\psi = 2\phi$ it follows that*

$$0 \leq F(-\lambda') - F(0) - F(\lambda_1 - \lambda') + \operatorname{Re}\{F(-\lambda' + i\mu') - F(i\mu') - F(\lambda_1 - \lambda' + i\mu')\} + f(0)\psi + \epsilon.$$

Proof. (a) In (8.1), choose $\chi = \chi_* = \chi_1, \rho = \rho'$ and $\rho_* = \rho_1$ with $\sigma = \beta'$ in (8.1) giving

$$(8.3) \quad 0 \leq \mathcal{K}(\beta', \chi_0) + \mathcal{K}(\beta' + i\gamma', \chi_1) + \mathcal{K}(\beta', \chi_1) + \mathcal{K}(\beta' + i\gamma', \chi_0).$$

Apply Lemma 6.3 to each $\mathcal{K}(*, *)$ term and extract the relevant zeros as follows:

- For $\mathcal{K}(\beta', \chi_0)$ and $\mathcal{K}(\beta' + i\gamma', \chi_0)$, extract no zeros since by assumption $\lambda' \leq \lambda_2$ yielding

$$(8.4) \quad \begin{aligned} \mathcal{L}^{-1}\mathcal{K}(\beta', \chi_0) &\leq f(0)\phi\frac{\mathcal{L}_0}{\mathcal{L}} + F(-\lambda') + \epsilon, \\ \mathcal{L}^{-1}\mathcal{K}(\beta' + i\gamma', \chi_0) &\leq f(0)\phi\frac{\mathcal{L}_0}{\mathcal{L}} + \operatorname{Re}\{F(-\lambda' + i\mu')\} + \epsilon. \end{aligned}$$

- For $\mathcal{K}(\beta' + i\gamma', \chi_1)$ and $\mathcal{K}(\beta', \chi_1)$, extract $\{\rho_1, \rho'\}$ implying

$$(8.5) \quad \begin{aligned} \mathcal{L}^{-1}\mathcal{K}(\beta' + i\gamma', \chi_1) &\leq f(0)\phi\frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} - F(0) - \operatorname{Re}\{F(\lambda_1 - \lambda' + i\mu')\} + \epsilon, \\ \mathcal{L}^{-1}\mathcal{K}(\beta', \chi_1) &\leq f(0)\phi\frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} - F(\lambda_1 - \lambda') - \operatorname{Re}\{F(i\mu')\} + \epsilon. \end{aligned}$$

Using (8.4) and (8.5) in (8.3) and rescaling ϵ , the desired inequality follows from except with $\psi = \phi \cdot \frac{2\mathcal{L}_0 + 2\mathcal{L}_{\chi_1}}{\mathcal{L}}$. From Lemma 3.1, $\psi \leq 4\phi$ so we may use $\psi = 4\phi$ instead.

(b) Use (8.2) with $\chi = \chi_0$, $\sigma = \beta'$ and $\rho = \rho'$, from which we deduce

$$0 \leq \mathcal{K}(\beta', \chi_0) + \mathcal{K}(\beta' + i\gamma', \chi_0) \quad \text{for } \sigma > 0.$$

Similar to (a), for both $\mathcal{K}(*, \chi_0)$ terms, apply Lemma 6.3 extracting both zeros $\{\rho_1, \rho'\}$ yielding

$$\begin{aligned} \mathcal{L}^{-1}\mathcal{K}(\beta', \chi_0) &\leq f(0)\phi\frac{\mathcal{L}_0}{\mathcal{L}} + F(-\lambda') - F(\lambda_1 - \lambda') - \operatorname{Re}\{F(i\mu')\} + \epsilon \\ \mathcal{L}^{-1}\mathcal{K}(\beta' + i\gamma', \chi_0) &\leq f(0)\phi\frac{\mathcal{L}_0}{\mathcal{L}} + \operatorname{Re}\{F(-\lambda' + i\mu') - F(\lambda_1 - \lambda' + i\mu')\} - F(0) + \epsilon \end{aligned}$$

Combined with the previous inequality, this yields the desired result with $\psi = 2\phi \cdot \frac{\mathcal{L}_0}{\mathcal{L}}$. By Lemma 3.1, we may use $\psi = 2\phi$ instead. \square

8.1.1. λ_1 *very small*. We now obtain a preliminary version of the Deuring-Heilbronn phenomenon for zeros of $L(s, \chi_1)$.

Lemma 8.2. *Assume χ_1 and ρ_1 are real. Let $\epsilon > 0$ and suppose \mathcal{L} is sufficiently large depending on ϵ .*

(a) *If χ_1 is quadratic and $\lambda' \leq \lambda_2$, then either $\lambda' < 4e$ or*

$$\lambda' \geq \left(\frac{1}{2} - \epsilon\right) \log(\lambda_1^{-1}),$$

which is non-trivial for $\lambda_1 \leq 3.5 \times 10^{-10}$.

(b) *If χ_1 is principal, then either $\lambda' < 4e$ or*

$$\lambda' \geq (1 - \epsilon) \log(\lambda_1^{-1}),$$

which is non-trivial for $\lambda_1 \leq 1.8 \times 10^{-5}$.

Proof. The proof is a close adaptation of [HB95, p. 37]. From Lemma 6.4 and Lemma 8.1, we have that

$$0 \leq 2F(-\lambda') - F(0) - 2F(\lambda_1 - \lambda') + f(0)(\psi + \epsilon).$$

where ψ depends on the cases in Lemma 8.1 and we assume $f(0) > 0$. As in [HB95, p. 37], choose

$$f(t) = \begin{cases} x_0 - t & 0 \leq t \leq x_0 \\ 0 & t \geq x_0 \end{cases}$$

for which Conditions 1 and 2 hold. Then by the same calculations, we see that

$$2F(-\lambda') - 2F(\lambda_1 - \lambda') \leq \frac{2x_0\lambda_1 \exp(x_0\lambda')}{(\lambda')^2}, \quad F(0) = \frac{1}{2}x_0^2, \quad f(0) = x_0,$$

and so from the first inequality, we have that

$$2x_0\lambda_1(\lambda')^{-2} \exp(x_0\lambda') - \frac{1}{2}x_0^2 + x_0(\psi + \epsilon) \geq 0.$$

Choose $x_0 := 2\psi + \frac{1}{\lambda'} + 2\epsilon$ so that the dependence on f is uniform for $\lambda' \geq 1$. With this choice, our inequality above then leads to

$$\lambda_1 \geq \frac{\lambda'}{4} \exp(-x_0\lambda') = \frac{\lambda'}{4e} \exp(-(2\psi + 2\epsilon)\lambda').$$

When $\lambda' \geq 4e$, we conclude

$$\lambda' \geq (\frac{1}{2\psi} - \epsilon) \log(\lambda_1^{-1}).$$

The result in each case follows from the value of ψ given in Lemma 8.1 and noting $\phi = 1/4$. \square

8.1.2. λ_1 *small*. Here we create a “numerical version” of Lemma 8.1.

Lemma 8.3. *Let $\epsilon > 0$ and for $b \geq 0$, assume $0 < \lambda_1 \leq b$ and retain the assumptions of Lemma 8.1. Suppose, for some $\lambda'_b > 0$, we have*

$$(8.6) \quad 2F(-\lambda'_b) - 2F(b - \lambda'_b) - F(0) + f(0)\psi \leq 0$$

where $\psi = 4\phi$ or 2ϕ if χ_1 is quadratic or principal respectively. Then $\lambda' \geq \lambda'_b - \epsilon$ for \mathcal{L} sufficiently large depending on ϵ, b and f .

Proof. Lemma 8.1 and Lemma 6.4 imply that

$$0 \leq 2F(-\lambda') - 2F(\lambda_1 - \lambda') - F(0) + f(0)\psi + \epsilon.$$

Now, by Conditions 1 and 2, the function

$$F(-\lambda) - F(b - \lambda) = \int_0^\infty f(t)e^\lambda(1 - e^{-b})dt$$

is an increasing function of λ and also of b . Hence, the previous inequality implies that

$$0 \leq 2F(-\lambda') - 2F(b - \lambda') - F(0) + f(0)\psi + \epsilon.$$

On the other hand, from the increasing behaviour of $F(-\lambda) - F(b - \lambda)$, we may deduce that, if (8.6) holds for some λ'_b , then

$$0 \leq 2F(-\lambda) - 2F(b - \lambda) - F(0) + f(0)\psi \quad \text{only if } \lambda \geq \lambda'_b.$$

Comparing with the previous inequality and choosing a new value of ϵ , we conclude $\lambda' \geq \lambda'_b - \epsilon$. See [KN12, p.773] for details on this last argument. \square

In each case, employing Lemma 8.3 for various values of b requires a choice of f depending on b which maximizes the computed value of λ'_b . Based on numerical experimentation, we choose $f = f_\lambda$ from [HB95, Lemma 7.2] with parameter $\lambda = \lambda(b)$. This produces Tables 2 and 3. Note that the bounds in Table 2 are applicable in a later subsection for bounds on λ_2 .

8.1.3. λ_1 *medium*. As a first attempt, we use techniques similar to before.

Lemma 8.4. *Assume χ_1 and ρ_1 are real. Provided \mathcal{L} is sufficiently large, it follows that if ρ' is real then*

$$\lambda' \geq \begin{cases} 0.6069 & \text{if } \chi_1 \text{ is quadratic and } \lambda' \leq \lambda_2, \\ 1.2138 & \text{if } \chi_1 \text{ is principal,} \end{cases}$$

and if ρ' is complex then

$$\lambda' \geq \begin{cases} 0.1722 & \text{if } \chi_1 \text{ is quadratic and } \lambda' \leq \lambda_2, \\ 0.3444 & \text{if } \chi_1 \text{ is principal.} \end{cases}$$

$\lambda_1 \leq$	$\frac{1}{2} \log \lambda_1^{-1} \geq$	$\lambda^* \geq$	λ	$\lambda_1 \leq$	$\frac{1}{2} \log \lambda_1^{-1} \geq$	$\lambda^* \geq$	λ
10^{-10}	11.51	10.99	.8010	.030	1.753	1.137	.6183
10^{-9}	10.36	9.920	.7975	.035	1.676	1.048	.6092
10^{-8}	9.210	8.838	.7930	.040	1.609	.9699	.6007
10^{-7}	8.059	7.740	.7873	.045	1.551	.9016	.5927
10^{-6}	6.908	6.623	.7796	.050	1.498	.8407	.5852
10^{-5}	5.756	5.481	.7687	.055	1.450	.7859	.5780
10^{-4}	4.605	4.303	.7521	.060	1.407	.7362	.5711
.001	3.454	3.075	.7239	.065	1.367	.6906	.5644
.005	2.649	2.176	.6896	.070	1.330	.6487	.5580
.010	2.303	1.778	.6679	.075	1.295	.6098	.5517
.015	2.100	1.542	.6522	.080	1.263	.5738	.5457
.020	1.956	1.374	.6394	.085	1.233	.5401	.5397
.025	1.844	1.244	.6283				

TABLE 2. Bounds for $\lambda^* = \lambda'$ with χ_1 quadratic, ρ_1 real and λ_1 small; and for $\lambda^* = \lambda_2$ with χ_1 quadratic, ρ_1 real, χ_2 principal and λ_1 small.

$\lambda_1 \leq$	$\log \lambda_1^{-1} \geq$	$\lambda' \geq$	λ	$\lambda_1 \leq$	$\log \lambda_1^{-1} \geq$	$\lambda' \geq$	λ
10^{-5}	11.51	11.66	1.545	.085	2.465	1.869	1.193
10^{-4}	9.210	9.324	1.516	.0875	2.436	1.836	1.189
.001	6.908	6.902	1.468	.090	2.408	1.803	1.185
.005	5.298	5.135	1.413	.095	2.354	1.741	1.178
.010	4.605	4.352	1.379	.100	2.303	1.681	1.170
.015	4.200	3.887	1.355	.105	2.254	1.625	1.163
.020	3.912	3.555	1.336	.110	2.207	1.572	1.156
.025	3.689	3.297	1.319	.115	2.163	1.521	1.149
.030	3.507	3.084	1.304	.120	2.120	1.472	1.142
.035	3.352	2.905	1.291	.125	2.079	1.426	1.135
.040	3.219	2.749	1.279	.130	2.040	1.381	1.129
.045	3.101	2.611	1.267	.135	2.002	1.338	1.122
.050	2.996	2.488	1.257	.140	1.966	1.297	1.116
.055	2.900	2.377	1.246	.145	1.931	1.258	1.110
.060	2.813	2.275	1.237	.150	1.897	1.220	1.103
.065	2.733	2.181	1.227	.155	1.864	1.183	1.097
.070	2.659	2.095	1.218	.160	1.833	1.148	1.091
.075	2.590	2.015	1.210	.165	1.802	1.113	1.085
.080	2.526	1.940	1.201	.170	1.772	1.080	1.079

TABLE 3. Bounds for λ' with χ_1 principal, ρ_1 real and λ_1 small.

Proof. If ρ' is real, then $\mu' = 0$. From Lemma 8.1 it follows that

$$0 \leq F(-\lambda') - F(0) - F(\lambda_1 - \lambda') + \frac{1}{2}f(0)\psi + \epsilon$$

where ϵ, f, ψ are specified in Lemma 8.1. Since F is decreasing by Condition 2,

$$0 \leq F(-\lambda') - 2F(0) + \frac{1}{2}f(0)\psi + \epsilon.$$

We select the function from [HB95, Lemma 7.5] corresponding to $k = 2$. Hence,

$$\frac{1}{\lambda'} \cos^2 \theta \leq \frac{1}{2} \psi + \epsilon.$$

For $k = 2$, we find $\theta = 0.9873\dots$ and so $\lambda' \geq \frac{0.6069}{\psi}$ for an appropriate choice of ϵ . If ρ' is complex, then we follow a similar argument selecting f from [HB95, Lemma 7.5] corresponding to $k = \frac{3}{2}$ (i.e. $\theta = 1.2729\dots$). \square

For ρ' complex, a method based on Section 5 leads to better bounds than Lemma 8.4.

Lemma 8.5. *Assume χ_1 and ρ_1 is real and also suppose ρ' is complex. Let $\lambda > 0$ and $J > 0$. If \mathcal{L} is sufficiently large depending on ϵ, λ and J then*

$$0 \leq (J^2 + \frac{1}{2}) \left(P_4(1) - P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) \right) - 2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) + \begin{cases} 2\phi(J+1)^2\lambda + \epsilon & \text{if } \chi_1 \text{ is quadratic,} \\ \phi(J+1)^2\lambda + \epsilon & \text{if } \chi_1 \text{ is principal,} \end{cases}$$

provided

$$(8.7) \quad \frac{J_0}{(\lambda + \lambda')^4} + \frac{1}{(\lambda + \lambda_1)^4} > \frac{1}{\lambda^4} \quad \text{with } J_0 = \min\left\{\frac{J}{2} + \frac{1}{2J}, 4J\right\}.$$

Remark. Recall $P_4(X)$ is a fixed polynomial throughout the paper and is defined by (5.6).

Proof. For an admissible polynomial $P(X) = \sum_{k=1}^d a_k X^k$, we begin with the inequality

$$\begin{aligned} 0 &\leq \chi_0(\mathbf{n})(1 + \chi_1(\mathbf{n}))(J + \operatorname{Re}\{\chi_1(\mathbf{n})(N\mathbf{n})^{-i\gamma'}\})^2 \\ &= (J^2 + \frac{1}{2})(\chi_0(\mathbf{n}) + \chi_1(\mathbf{n})) + 2J \cdot (\operatorname{Re}\{\chi_0(\mathbf{n})(N\mathbf{n})^{-i\gamma'}\} + \operatorname{Re}\{\chi_1(\mathbf{n})(N\mathbf{n})^{-i\gamma'}\}) \\ &\quad + \frac{1}{2} \cdot (\operatorname{Re}\{\chi_0(\mathbf{n})(N\mathbf{n})^{-2i\gamma'}\} + \operatorname{Re}\{\chi_1(\mathbf{n})(N\mathbf{n})^{-2i\gamma'}\}). \end{aligned}$$

To introduce $\mathcal{P}(s, \chi) = \mathcal{P}(s, \chi; P)$, we multiply the above inequality by

$$\frac{\Lambda(\mathbf{n})}{(N\mathbf{n})^\sigma} \left(\sum_{k=1}^d a_k \frac{((\sigma - 1) \log N\mathbf{n})^{k-1}}{(k-1)!} \right)$$

with $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$ and sum over ideals \mathbf{n} yielding

$$(8.8) \quad \begin{aligned} 0 &\leq (J^2 + \frac{1}{2})(\mathcal{P}(\sigma, \chi_0) + \mathcal{P}(\sigma, \chi_1)) + 2J \cdot (\mathcal{P}(\sigma + i\gamma', \chi_0) + \mathcal{P}(\sigma + i\gamma', \chi_1)) \\ &\quad + \frac{1}{2} \cdot (\mathcal{P}(\sigma + 2i\gamma', \chi_0) + \mathcal{P}(\sigma + 2i\gamma', \chi_1)). \end{aligned}$$

Taking $P(X) = P_4(X)$ so $a_1 = 1$, we consider the two cases depending on χ_1 .

(a) χ_1 is quadratic: Apply Lemma 5.4 to each $\mathcal{P}(*, *)$ term in (8.8) extracting the pole from χ_0 -terms and the zeros ρ_1, ρ' (and possibly $\overline{\rho'}$) from the χ_1 -terms. Each of these applications

yields the following:

$$\begin{aligned}
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma, \chi_0) &\leq \phi \frac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{P_4(1)}{\lambda}, \\
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma, \chi_1) &\leq \phi \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + \epsilon - \frac{1}{\lambda} \cdot \left(P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) + \operatorname{Re}\left\{P_4\left(\frac{\lambda}{\lambda + \lambda' + i\mu'}\right)\right\} \right), \\
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma + i\gamma', \chi_0) &\leq \phi \frac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda} \cdot \operatorname{Re}\left\{P_4\left(\frac{\lambda}{\lambda + i\mu'}\right)\right\}, \\
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma + i\gamma', \chi_1) &\leq \phi \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + \epsilon - \frac{1}{\lambda} \cdot \left(P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) + \operatorname{Re}\left\{P_4\left(\frac{\lambda}{\lambda + \lambda_1 + i\mu'}\right) + P_4\left(\frac{\lambda}{\lambda + \lambda' + 2i\mu'}\right)\right\} \right), \\
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma + 2i\gamma', \chi_0) &\leq \phi \frac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda} \cdot \operatorname{Re}\left\{P_4\left(\frac{\lambda}{\lambda + 2i\mu'}\right)\right\}, \\
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma + 2i\gamma', \chi_1) &\leq \phi \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + \epsilon - \frac{1}{\lambda} \cdot \operatorname{Re}\left\{P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 2i\mu'}\right) + P_4\left(\frac{\lambda}{\lambda + \lambda' + i\mu'}\right)\right\},
\end{aligned}$$

provided \mathcal{L} is sufficiently large depending on ϵ and λ . For the term $\mathcal{P}(\sigma + i\gamma', \chi_1)$ we extracted all 3 zeros of χ_1 , i.e. that $\mu' \neq 0$. Substituting into (8.8) and noting $\frac{\mathcal{L}_0 + \mathcal{L}_{\chi_1}}{\mathcal{L}} \leq 2$ by Lemma 3.1, we find

$$(8.9) \quad 0 \leq (J^2 + \tfrac{1}{2}) \left(P_4(1) - P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) \right) - 2JP_4\left(\frac{\lambda}{\lambda + \lambda'}\right) - A - B + 2\phi(J+1)^2\lambda + \epsilon$$

where

$$\begin{aligned}
A &= \operatorname{Re}\left\{ (J^2 + 1) \cdot P_4\left(\frac{\lambda}{\lambda + \lambda' + i\mu'}\right) \right\} + 2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + i\mu'}\right) - 2J \cdot P_4\left(\frac{\lambda}{\lambda + i\mu'}\right), \\
B &= \operatorname{Re}\left\{ 2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda' + 2i\mu'}\right) + \frac{1}{2} \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 2i\mu'}\right) - \frac{1}{2} \cdot P_4\left(\frac{\lambda}{\lambda + 2i\mu'}\right) \right\}.
\end{aligned}$$

From Lemma 5.6, we see that $A, B \geq 0$ provided

$$\frac{J^2 + 1}{(\lambda + \lambda')^4} + \frac{2J}{(\lambda + \lambda_1)^4} > \frac{2J}{\lambda^4} \quad \text{and} \quad \frac{2J}{(\lambda + \lambda')^4} + \frac{1/2}{(\lambda + \lambda_1)^4} > \frac{1/2}{\lambda^4}.$$

Assumption (8.7) implies both of these inequalities.

(b) χ_1 is principal: Then (8.8) becomes

$$0 \leq (2J^2 + 1)\mathcal{P}(\sigma, \chi_0) + 4J \cdot \mathcal{P}(\sigma + i\gamma', \chi_0) + \mathcal{P}(\sigma + 2i\gamma', \chi_0).$$

We similarly apply Lemma 5.4 to each term above extracting the pole and zeros ρ_1, ρ' (and possibly $\bar{\rho}$). Each of these applications yields the following:

$$\begin{aligned}
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma, \chi_0) &\leq \phi \frac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda} \left(P_4(1) - P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) + \operatorname{Re}\left\{P_4\left(\frac{\lambda}{\lambda + \lambda' + i\mu'}\right)\right\} \right), \\
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma + i\gamma', \chi_0) &\leq \phi \frac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda} \left(-P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) + \operatorname{Re}\left\{P_4\left(\frac{\lambda}{\lambda + i\mu'}\right) - P_4\left(\frac{\lambda}{\lambda + \lambda_1 + i\mu'}\right) \right. \right. \\
&\quad \left. \left. - P_4\left(\frac{\lambda}{\lambda + \lambda' + 2i\mu'}\right)\right\} \right), \\
\mathcal{L}^{-1} \cdot \mathcal{P}(\sigma + 2i\gamma', \chi_0) &\leq \phi \frac{\mathcal{L}_0}{\mathcal{L}} + \epsilon + \frac{1}{\lambda} \cdot \operatorname{Re}\left\{P_4\left(\frac{\lambda}{\lambda + 2i\mu'}\right) - P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 2i\mu'}\right) - P_4\left(\frac{\lambda}{\lambda + \lambda' + i\mu'}\right)\right\}.
\end{aligned}$$

$\lambda_1 \leq$	$\frac{1}{2} \log(1/\lambda_1)$	$\lambda^* \geq$	λ	J	$\lambda_1 \leq$	$\frac{1}{2} \log(1/\lambda_1)$	$\lambda^* \geq$	λ	J
.09	1.204	.5261	1.239	.8837	.19	.8304	.3759	1.417	.8483
.10	1.151	.5063	1.265	.8793	.20	.8047	.3649	1.428	.8454
.11	1.104	.4880	1.289	.8752	.21	.7803	.3544	1.438	.8426
.12	1.060	.4709	1.310	.8714	.22	.7571	.3443	1.447	.8398
.1227	1.049	.4665	1.316	.8704	.23	.7348	.3347	1.455	.8370
.13	1.020	.4549	1.330	.8677	.24	.7136	.3254	1.463	.8343
.14	.9831	.4398	1.348	.8642	.25	.6931	.3165	1.471	.8316
.15	.9486	.4257	1.364	.8608	.26	.6735	.3080	1.477	.8289
.16	.9163	.4122	1.379	.8575	.27	.6547	.2998	1.483	.8263
.17	.8860	.3995	1.393	.8544	.28	.6365	.2918	1.489	.8237
.18	.8574	.3874	1.405	.8513	.2866	.6248	.2868	1.493	.8220

TABLE 4. Bounds for $\lambda^* = \lambda'$ with χ_1 quadratic, ρ_1 real and λ_1 medium;
and for $\lambda^* = \lambda_2$ with χ_1 quadratic, ρ_1 real, χ_2 principal, and ρ_2 complex.

Substituting these into the previous inequality, noting $\frac{\mathcal{L}_0}{\mathcal{L}} \leq 1$ and dividing by 2, we obtain (8.9) except with 2ϕ replaced by ϕ . Following the same argument, we obtain the desired result. \square

Again, we exhibit a “numerical version” of Lemma 8.5.

Corollary 8.6. *Assume χ_1 and ρ_1 is real and suppose ρ' is complex. Let $\epsilon > 0$. Suppose $0 < \lambda_1 \leq b, \lambda > 0, J > 0$ and that there exists $\lambda'_b \in [0, \infty)$ satisfying*

$$(J^2 + \tfrac{1}{2})(P_4(1) - P_4(\frac{\lambda}{\lambda + b})) - 2J \cdot P_4(\frac{\lambda}{\lambda + \lambda'_b}) + \psi(J + 1)^2\lambda \leq 0$$

where $\psi = 2\phi$ or ϕ if χ_1 is quadratic or principal respectively. Then it follows that $\lambda' \geq \lambda'_b - \epsilon$ for \mathcal{L} sufficiently large depending on ϵ, λ and J provided

$$\frac{J_0}{(\lambda + \lambda'_b)^4} + \frac{1}{(\lambda + b)^4} > \frac{1}{\lambda^4} \quad \text{where } J_0 = \min\{\frac{J}{2} + \frac{1}{2J}, 4J\}.$$

Proof. From Lemma 8.5,

$$0 \leq (J^2 + \tfrac{1}{2})(P_4(1) - P_4(\frac{\lambda}{\lambda + \lambda_1})) - 2J \cdot P_4(\frac{\lambda}{\lambda + \lambda'}) + (2 - E_0(\chi_1)) \cdot \phi(J + 1)^2\lambda + \epsilon.$$

Since P_4 has non-negative coefficients and $P_4(0) = 0$, the above expression is *increasing* with λ_1 and λ' . From this observation, the desired result follows. \square

Corollary 8.6 gives lower bounds for λ' for certain ranges of λ_1 . For each range $0 < \lambda_1 \leq b$, we choose $\lambda = \lambda(b) > 0, J = J(b) > 0$ to produce an optimal lower bound λ'_b for λ' . This produces Tables 4 and 5.

8.1.4. *Summary of bounds on λ' .* We collect the results of the previous subsections for each range of λ_1 into a single result for ease of use.

Proposition 8.7. *Assume χ_1 and ρ_1 are real. Suppose \mathcal{L} is sufficiently large depending on $\epsilon > 0$. Then:*

$\lambda_1 \leq$	$\log(1/\lambda_1) \geq$	$\lambda' \geq$	λ	J	$\lambda_1 \leq$	$\log(1/\lambda_1) \geq$	$\lambda' \geq$	λ	J
.18	1.715	1.052	2.478	.8837	.39	.9416	.7406	2.845	.8469
.19	1.661	1.032	2.505	.8815	.40	.9163	.7297	2.855	.8454
.20	1.609	1.013	2.530	.8793	.41	.8916	.7191	2.866	.8440
.21	1.561	.9939	2.555	.8772	.42	.8675	.7087	2.875	.8426
.22	1.514	.9759	2.578	.8752	.43	.8440	.6985	2.885	.8412
.23	1.470	.9586	2.600	.8733	.44	.8210	.6886	2.894	.8398
.24	1.427	.9418	2.621	.8714	.45	.7985	.6788	2.903	.8384
.25	1.386	.9255	2.641	.8695	.46	.7765	.6693	2.911	.8370
.26	1.347	.9098	2.660	.8677	.47	.7550	.6600	2.919	.8356
.27	1.309	.8945	2.678	.8659	.48	.7340	.6508	2.927	.8343
.28	1.273	.8797	2.695	.8642	.49	.7133	.6418	2.934	.8329
.29	1.238	.8653	2.712	.8625	.50	.6931	.6330	2.941	.8316
.30	1.204	.8513	2.728	.8608	.51	.6733	.6244	2.948	.8303
.31	1.171	.8377	2.743	.8592	.52	.6539	.6159	2.955	.8289
.32	1.139	.8245	2.758	.8575	.53	.6349	.6076	2.961	.8276
.33	1.109	.8116	2.772	.8560	.54	.6162	.5995	2.967	.8263
.34	1.079	.7990	2.785	.8544	.55	.5978	.5915	2.973	.8250
.35	1.050	.7867	2.798	.8528	.56	.5798	.5837	2.978	.8237
.36	1.022	.7748	2.811	.8513	.57	.5621	.5760	2.984	.8224
.37	.9943	.7631	2.822	.8498	.5733	.5563	.5735	2.985	.8220
.38	.9676	.7517	2.834	.8483					

TABLE 5. Bounds for λ' with χ_1 principal, ρ_1 real and λ_1 medium.

(a) Suppose χ_1 is quadratic and $\lambda' \leq \lambda_2$. Then

$$\lambda' \geq \begin{cases} (\frac{1}{2} - \epsilon) \log \lambda_1^{-1} & \text{if } \lambda_1 \leq 10^{-10} \\ 0.2103 \log \lambda_1^{-1} & \text{if } \lambda_1 \leq 0.1227 \end{cases}$$

and if $\lambda_1 > 0.1227$ then the bounds in Table 4 apply and $\lambda' \geq 0.2866$.

(b) Suppose χ_1 is principal. If $\lambda_1 \leq 0.0875$, then

$$\lambda' \geq \begin{cases} (1 - \epsilon) \log \lambda_1^{-1} & \text{if } \lambda_1 \leq 10^{-5} \\ 0.7399 \log \lambda_1^{-1} & \text{if } \lambda_1 \leq 0.0875 \end{cases}$$

and if $\lambda_1 > 0.0875$ then the bounds in Tables 3 and 5 apply and $\lambda' \geq 0.5733$.

Remark. The constants 0.1227 and 0.0875 come a posteriori from the corresponding zero-free regions established in Section 10.

Proof. (a) Suppose $\lambda_1 \leq 10^{-10}$. From Table 2, we see that $\lambda' \geq 10.99 > 4e$ and so the desired bound follows from Lemma 8.2. Suppose $\lambda_1 \leq 0.1227$. One compares Lemma 8.4 and Table 4 and finds that the latter gives weaker bounds. Thus, we only consider Tables 2 and 4 for this range of λ_1 . For the subinterval $\lambda_1 \in [0.12, 0.1227]$, it follows that

$$\lambda' \geq 0.4663 \geq \frac{0.4663}{\log 1/0.12} \log \lambda_1^{-1} \geq 0.2200 \log \lambda_1^{-1}.$$

Repeat this process for each subinterval $[10^{-10}, 10^{-9}]$, $[10^{-9}, 10^{-8}]$, \dots , $[0.85, 0.9]$, \dots , $[0.12, 0.1227]$ to obtain the desired bound. For $\lambda_1 > 0.1227$, one again compares Lemma 8.4 and Table 4 and finds that the latter gives weaker bounds. For (b), we argue analogous to (a) except we only use Table 3 for $\lambda_1 \leq 0.0875$. \square

8.2. Bounds for λ_2 . We follow the same general approach as λ' with natural modifications. Throughout, we shall assume $\lambda_2 \leq \lambda'$; otherwise, we may use the bounds from Section 8.1 on λ' .

Lemma 8.8. *Assume χ_1 and ρ_1 are real and also that $\lambda_2 \leq \lambda'$. Suppose f satisfies Conditions 1 and 2. For $\epsilon > 0$, provided \mathcal{L} is sufficiently large depending on ϵ and f , the following holds:*

(a) *If χ_1, χ_2 are non-principal, then with $\psi = 4\phi$ it follows that*

$$0 \leq F(-\lambda_2) - F(0) - F(\lambda_1 - \lambda_2) + f(0)\psi + \epsilon.$$

(b) *If χ_1 is principal, then χ_2 is necessarily non-principal and with $\psi = 2\phi$ it follows that*

$$0 \leq F(-\lambda_2) - F(0) - F(\lambda_1 - \lambda_2) + f(0)\psi + \epsilon.$$

(c) *If χ_2 is principal, then χ_1 is necessarily non-principal and with $\psi = 4\phi$ it follows that*

$$0 \leq F(-\lambda_2) - F(0) - F(\lambda_1 - \lambda_2) + \operatorname{Re}\{F(-\lambda_2 + i\mu_2) - F(i\mu_2) - F(\lambda_1 - \lambda_2 + i\mu_2)\} + f(0)\psi + \epsilon.$$

Proof. In (8.1), set $(\chi, \rho) = (\chi_1, \rho_1)$ and $(\chi_*, \rho_*) = (\chi_2, \rho_2)$ and $\sigma = \beta_2$, which gives (8.10)

$$0 \leq \mathcal{K}(\beta_2, \chi_0) + \mathcal{K}(\beta_2, \chi_1) + \mathcal{K}(\beta_2 + i\gamma_2, \chi_2) + \frac{1}{2}\mathcal{K}(\beta_2 + i\gamma_2, \chi_1\chi_2) + \frac{1}{2}\mathcal{K}(\beta_2 - i\gamma_2, \chi_1\overline{\chi_2}).$$

The arguments involved are entirely analogous to Lemma 8.1 so we omit the details here. For all cases, one applies Lemma 6.3 to each $\mathcal{K}(*, *)$ term, extracting ρ_1 or ρ_2 whenever possible. We remark that $\chi_1\chi_2$ and $\chi_1\overline{\chi_2}$ are always non-principal by construction (see Section 3). \square

8.2.1. λ_1 very small. We include the final result here without proof for the sake of brevity.

Lemma 8.9. *Assume χ_1 and ρ_1 are real and $\lambda_2 \leq \lambda'$. Suppose \mathcal{L} is sufficiently large depending on $\epsilon > 0$.*

(a) *If χ_1, χ_2 are non-principal, then either $\lambda_2 < 2e$ or*

$$\lambda_2 \geq \left(\frac{1}{2} - \epsilon\right) \log(\lambda_1^{-1}),$$

which is non-trivial for $\lambda_1 \leq 1.8 \times 10^{-5}$.

(b) *If χ_1 is principal, then χ_2 is necessarily non-principal and either $\lambda_2 < 2e$ or*

$$\lambda_2 \geq (1 - \epsilon) \log(\lambda_1^{-1}),$$

which is non-trivial for $\lambda_1 \leq 4.3 \times 10^{-3}$.

(c) *If χ_2 is principal, then χ_1 is necessarily non-principal and either $\lambda_2 < 4e$ or*

$$\lambda_2 \geq \left(\frac{1}{2} - \epsilon\right) \log(\lambda_1^{-1}),$$

which is non-trivial for $\lambda_1 \leq 3.5 \times 10^{-10}$.

Proof. Analogous to Lemma 8.2 using Lemma 8.8 in place of Lemma 8.1. We omit the details for brevity. \square

8.2.2. λ_1 small.

Lemma 8.10. *Assume χ_1 and ρ_1 are real and also that $\lambda_2 \leq \lambda'$. Suppose f satisfies Conditions 1 and 2. Let $\epsilon > 0$ and assume $0 < \lambda_1 \leq b$ for some $b > 0$. Suppose, for some $\tilde{\lambda}_b > 0$, we have*

$$F(-\tilde{\lambda}_b) - F(b - \tilde{\lambda}_b) - F(0) + 4\phi f(0) \leq 0 \quad \text{if } \chi_1, \chi_2 \text{ are non-principal,}$$

$$F(-\tilde{\lambda}_b) - F(b - \tilde{\lambda}_b) - F(0) + 2\phi f(0) \leq 0 \quad \text{if } \chi_1 \text{ is principal,}$$

$$2F(-\tilde{\lambda}_b) - 2F(b - \tilde{\lambda}_b) - F(0) + 4\phi f(0) \leq 0 \quad \text{if } \chi_2 \text{ is principal}$$

Then, according to the above cases, $\lambda_2 \geq \tilde{\lambda}_b - \epsilon$ provided \mathcal{L} is sufficiently large depending on ϵ, b and f .

Proof. Analogous to Lemma 8.3 using Lemma 8.8 in place of Lemma 8.1. Hence, we omit the proof. \square

As before, Lemma 8.10 requires a choice of f depending on b which maximizes the computed value of $\tilde{\lambda}_b$. Based on numerical experimentation, we choose $f = f_\lambda$ from [HB95, Lemma 7.2] with parameter $\lambda = \lambda(b)$ for all cases. This produces Tables 2, 6 and 7.

8.2.3. λ_1 medium. We first deal with the case when ρ_2 is real and χ_2 is principal, i.e. $\mu_2 = 0$.

Lemma 8.11. *Assume χ_1 and ρ_1 are real. Suppose \mathcal{L} is sufficiently large. If ρ_2 is real, then*

$$\lambda_2 \geq \begin{cases} 0.3034 & \text{if } \chi_1, \chi_2 \text{ are non-principal,} \\ 0.6069 & \text{otherwise.} \end{cases}$$

If ρ_2 is complex, then

$$\lambda_2 \geq \begin{cases} 0.3034 & \text{if } \chi_1, \chi_2 \text{ are non-principal,} \\ 0.6069 & \text{if } \chi_1 \text{ is principal,} \\ 0.1722 & \text{if } \chi_2 \text{ is principal.} \end{cases}$$

Proof. Analogous to Lemma 8.4 using Lemma 8.8 in place of Lemma 8.1. The arguments lead to selecting f from [HB95, Lemma 7.5] corresponding to $k = 2$ (i.e. $\theta = 0.9873\dots$) when ρ_2 is real or χ_2 is non-principal, and to $k = 3/2$ (i.e. $\theta = 1.2729\dots$) when ρ_2 is complex and χ_2 is principal. \square

For χ_2 principal and ρ_2 complex, the “polynomial method” of Section 5 yields better bounds.

Lemma 8.12. *Assume χ_1 is quadratic and ρ_1 is real. Further suppose χ_2 is principal and ρ_2 is complex. Let $\lambda > 0$ and $J > 0$. If \mathcal{L} is sufficiently large depending on ϵ, λ and J , then*

$$0 \leq (J^2 + \tfrac{1}{2})(P_4(1) - P_4(\tfrac{\lambda}{\lambda + \lambda_1})) - 2JP_4(\tfrac{\lambda}{\lambda + \lambda_2}) + 2\phi(J + 1)^2\lambda + \epsilon$$

provided

$$(8.11) \quad \frac{J_0}{(\lambda + \lambda_2)^4} + \frac{1}{(\lambda + \lambda_1)^4} > \frac{1}{\lambda^4}.$$

with $J_0 = \min\{\frac{J}{2} + \frac{1}{2J}, 4J\}$.

$\lambda_1 \leq$	$\frac{1}{2} \log \lambda_1^{-1} \geq$	$\lambda_2 \geq$	λ		$\lambda_1 \leq$	$\frac{1}{2} \log \lambda_1^{-1} \geq$	$\lambda_2 \geq$	λ
10^{-5}	5.756	5.828	.7725		.155	.9322	.6288	.5489
10^{-4}	4.605	4.662	.7579		.160	.9163	.6122	.5459
.001	3.454	3.451	.7342		.165	.9009	.5962	.5429
.005	2.649	2.569	.7065		.170	.8860	.5808	.5400
.010	2.303	2.178	.6896		.175	.8715	.5659	.5371
.015	2.100	1.947	.6776		.180	.8574	.5515	.5342
.020	1.956	1.783	.6679		.185	.8437	.5376	.5314
.025	1.844	1.654	.6596		.190	.8304	.5242	.5286
.030	1.753	1.550	.6522		.195	.8174	.5111	.5258
.035	1.676	1.461	.6455		.200	.8047	.4985	.5231
.040	1.609	1.384	.6394		.205	.7924	.4863	.5203
.045	1.551	1.317	.6337		.210	.7803	.4744	.5176
.050	1.498	1.256	.6283		.215	.7686	.4629	.5150
.055	1.450	1.202	.6232		.220	.7571	.4517	.5123
.060	1.407	1.152	.6183		.225	.7458	.4408	.5097
.065	1.367	1.107	.6137		.230	.7348	.4302	.5070
.070	1.330	1.065	.6092		.235	.7241	.4200	.5044
.075	1.295	1.026	.6049		.240	.7136	.4100	.5018
.080	1.263	.9895	.6007		.245	.7032	.4002	.4993
.085	1.233	.9555	.5967		.250	.6931	.3908	.4967
.090	1.204	.9236	.5928		.255	.6832	.3816	.4942
.095	1.177	.8935	.5890		.260	.6735	.3726	.4916
.100	1.151	.8652	.5853		.265	.6640	.3638	.4891
.105	1.127	.8383	.5816		.270	.6547	.3553	.4866
.110	1.104	.8127	.5781		.275	.6455	.3470	.4841
.115	1.081	.7884	.5746		.280	.6365	.3389	.4817
.120	1.060	.7653	.5712		.285	.6276	.3310	.4792
.125	1.040	.7432	.5679		.290	.6189	.3233	.4768
.130	1.020	.7221	.5646		.295	.6104	.3158	.4743
.135	1.001	.7019	.5613		.300	.6020	.3084	.4719
.140	.9831	.6825	.5582					
.145	.9655	.6639	.5550					
.150	.9486	.6460	.5520					

TABLE 6. Bounds for λ_2 with χ_1 quadratic, ρ_1 real, χ_2 non-principal and λ_1 small.

Proof. This is analogous to Lemma 8.5 so we give a brief outline here. We begin with the inequality

$$\begin{aligned}
0 &\leq \chi_0(\mathbf{n})(1 + \chi_1(\mathbf{n}))(J + \operatorname{Re}\{(\mathbf{Nn})^{-i\gamma_2}\})^2 \\
&= (J^2 + \tfrac{1}{2})(\chi_0(\mathbf{n}) + \chi_1(\mathbf{n})) + 2J \cdot (\operatorname{Re}\{(\mathbf{Nn})^{-i\gamma_2}\} + \operatorname{Re}\{\chi_1(\mathbf{n})(\mathbf{Nn})^{-i\gamma_2}\}) \\
&\quad + \tfrac{1}{2} \cdot (\operatorname{Re}\{(\mathbf{Nn})^{-2i\gamma_2}\} + \operatorname{Re}\{\chi_1(\mathbf{n})(\mathbf{Nn})^{-2i\gamma_2}\}).
\end{aligned}$$

$\lambda_1 \leq$	$\log \lambda_1^{-1} \geq$	$\lambda_2 \geq$	λ	$\lambda_1 \leq$	$\log \lambda_1^{-1} \geq$	$\lambda_2 \geq$	λ
.004	5.521	6.150	1.448	.24	1.427	1.531	1.142
.006	5.116	5.705	1.434	.26	1.347	1.444	1.129
.008	4.828	5.386	1.422	.28	1.273	1.365	1.116
.010	4.605	5.137	1.413	.30	1.204	1.292	1.104
.015	4.200	4.682	1.394	.32	1.139	1.224	1.092
.020	3.912	4.357	1.379	.34	1.079	1.162	1.080
.025	3.689	4.103	1.366	.36	1.022	1.103	1.068
.03	3.507	3.895	1.355	.38	.9676	1.048	1.057
.04	3.219	3.565	1.336	.40	.9163	.9970	1.046
.05	2.996	3.309	1.319	.42	.8675	.9488	1.035
.06	2.813	3.099	1.304	.44	.8210	.9033	1.025
.07	2.659	2.922	1.291	.46	.7765	.8605	1.014
.08	2.526	2.769	1.279	.48	.7340	.8199	1.004
.0875	2.436	2.666	1.270	.50	.6931	.7816	.9934
.10	2.303	2.513	1.257	.52	.6539	.7452	.9833
.12	2.120	2.304	1.237	.54	.6162	.7106	.9733
.14	1.966	2.130	1.218	.56	.5798	.6778	.9633
.16	1.833	1.979	1.201	.58	.5447	.6466	.9535
.18	1.715	1.847	1.186	.60	.5108	.6168	.9438
.20	1.609	1.730	1.171	.6068	.4996	.6070	.9405
.22	1.514	1.625	1.156				

TABLE 7. Bounds for λ_2 with χ_1 principal, ρ_1 real and λ_1 small.

We introduce $\mathcal{P}(s, \chi) = \mathcal{P}(s, \chi; P_4)$ in the usual way with $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$, yielding

$$(8.12) \quad 0 \leq (J^2 + \frac{1}{2})(\mathcal{P}(\sigma, \chi_0) + \mathcal{P}(\sigma, \chi_1)) + 2J \cdot (\mathcal{P}(\sigma + i\gamma_2, \chi_0) + \mathcal{P}(\sigma + i\gamma_2, \chi_1)) \\ + \frac{1}{2} \cdot (\mathcal{P}(\sigma + 2i\gamma_2, \chi_0) + \mathcal{P}(\sigma + 2i\gamma_2, \chi_1)).$$

Next, apply Lemma 5.4 to each $\mathcal{P}(*, *)$ term in (8.12) extracting the zero ρ_2 from χ_0 -terms and the zero ρ_1 from the χ_1 -terms. One also extracts both zeros $\{\rho_2, \overline{\rho_2}\}$ from $\mathcal{P}(\sigma + i\gamma_2, \chi_0)$. Noting $\frac{\mathcal{L}_0 + \mathcal{L}_{\chi_1}}{\mathcal{L}} \leq 2$ by Lemma 3.1 and choosing a new ϵ , these applications yield the following:

$$(8.13) \quad 0 \leq (J^2 + \frac{1}{2})(P_4(1) - P_4(\frac{\lambda}{\lambda + \lambda_1})) - 2JP_4(\frac{\lambda}{\lambda + \lambda_2}) - A - B + 2\phi(J+1)^2\lambda + \epsilon$$

provided \mathcal{L} is sufficiently large depending on ϵ and λ and where

$$A = (J^2 + 1)\text{Re}\{P_4(\frac{\lambda}{\lambda + \lambda_2 + i\mu_2})\} + 2J \cdot \text{Re}\{P_4(\frac{\lambda}{\lambda + \lambda_1 + i\mu_2})\} - 2J \cdot \text{Re}\{P_4(\frac{\lambda}{\lambda + i\mu_2})\}, \\ B = 2J \cdot \text{Re}\{P_4(\frac{\lambda}{\lambda + \lambda_2 + 2i\mu_2})\} + \frac{1}{2} \cdot \text{Re}\{P_4(\frac{\lambda}{\lambda + \lambda_1 + 2i\mu_2})\} - \frac{1}{2} \cdot \text{Re}\{P_4(\frac{\lambda}{\lambda + 2i\mu_2})\}.$$

Assumption (8.11) implies $A, B \geq 0$ by Lemma 5.6 yielding the desired result from (8.13). \square

With an appropriate numerical version of Lemma 8.12, analogous to Corollary 8.6, we obtain lower bounds for λ_2 for $\lambda_1 \in [0, b]$ and fixed $b > 0$. Optimally choosing $\lambda = \lambda(b) > 0, J = J(b) > 0$ produces Table 4 again.

8.2.4. *Summary of bounds on λ_2 .* We collect the estimates of the previous subsections for each range of λ_1 into a one result for ease of use.

Proposition 8.13. *Assume χ_1 and ρ_1 are real. Suppose \mathcal{L} is sufficiently large depending on $\epsilon > 0$:*

(a) *Suppose χ_1 is quadratic and $\lambda_2 \leq \lambda'$. Then*

$$\lambda_2 \geq \begin{cases} (\frac{1}{2} - \epsilon) \log \lambda_1^{-1} & \text{if } \lambda_1 \leq 10^{-10} \\ 0.2103 \log \lambda_1^{-1} & \text{if } \lambda_1 \leq 0.1227 \end{cases}$$

and if $\lambda_1 > 0.1227$ then the bounds in Table 4 apply and $\lambda_2 \geq 0.2866$.

(b) *Suppose χ_1 is principal. Then*

$$\lambda_2 \geq (1 - \epsilon) \log \lambda_1^{-1} \quad \text{if } \lambda_1 \leq 0.0875.$$

and if $\lambda_1 > 0.0875$ then the bounds in Table 7 apply and $\lambda_2 \geq 0.6069$.

Remark. After comparing Propositions 8.7 and 8.13 in the case when χ_1 is quadratic, we realize that the additional assumptions $\lambda' \leq \lambda_2$ or $\lambda_2 \leq \lambda'$ are superfluous.

Proof. (a) First, suppose χ_2 is non-principal. For $\lambda_1 \leq 10^{-5}$, we see from Table 6 that $\lambda_2 \geq 5.828 > 2e$ so the desired bound follows from Lemma 8.9. For $10^{-5} \leq \lambda_1 \leq 0.1227$, consider Table 6. Apply the same process as in Proposition 8.7 to each subinterval $[10^{-5}, 10^{-4}], \dots, [0.12, 0.125]$ to obtain

$$\lambda_2 \geq 0.3506 \log \lambda_1^{-1}.$$

Now, suppose χ_2 is principal. For $\lambda_1 \leq 10^{-10}$, we see from Table 2 that $\lambda_2 \geq 10.99 > 4e$ so the desired bound follows from Lemma 8.9. For $10^{-10} \leq \lambda_1 \leq 0.1227$, consider Tables 2 and 4. Apply the same process as in Proposition 8.7 to each subinterval $[10^{-10}, 10^{-9}], \dots, [0.85, 0.9], \dots [0.12, 0.1227]$ and obtain

$$\lambda_2 \geq 0.2103 \log \lambda_1^{-1}.$$

Upon comparing the two cases, the latter gives weaker results in the range $\lambda_1 \leq 0.1227$. For $\lambda_1 > 0.1227$, we compare Lemma 8.11 and Tables 4 and 6 and see that last one gives the weakest bounds.

(b) Similar to (a) except we use Table 7 in conjunction with Lemma 8.9. The range $\lambda_1 \leq 0.004$ gives the bound $\lambda_2 \geq (1 - \epsilon) \log \lambda_1^{-1}$. The range $0.004 \leq \lambda_1 \leq 0.0875$ turns out to give a better bound but we opt to write a bound uniform for $\lambda_1 \leq 0.0875$. For $\lambda_1 > 0.0875$, we use Lemma 8.11 and Table 7. \square

9. ZERO REPULSION: χ_1 OR ρ_1 IS COMPLEX

When χ_1 or ρ_1 is complex, the effect of zero repulsion is lesser than when χ_1 and ρ_1 are real. Nonetheless we will follow the same general outline as the previous section, but with modified trigonometric identities and more frequently using the “polynomial method” of Section 5. Also, whether χ_1 is principal naturally affects our arguments in a significant manner so for clarity we further subdivide our results on this condition.

9.1. Bounds for λ' .

9.1.1. χ_1 non-principal.

Lemma 9.1. Assume χ_1 or ρ_1 is complex with χ_1 non-principal. Let $\lambda > 0, J \geq \frac{1}{4}$. If \mathcal{L} is sufficiently large depending on ϵ, λ and J then

$$0 \leq (J^2 + \frac{1}{2})P_4(1) - (J^2 + \frac{1}{2}) \cdot P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) - 2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) + 2(J+1)^2\phi\lambda + \epsilon$$

provided

$$(9.1) \quad \frac{J_0}{(\lambda + \lambda_1)^4} + \frac{1}{(\lambda + \lambda')^4} > \frac{1}{\lambda^4} \quad \text{with } J_0 = \min\{J + \frac{3}{4J}, 4J\}.$$

Proof. For simplicity, denote $\mathcal{P}(s, \chi) = \mathcal{P}(s, \chi; P_4)$. Our starting point is the trigonometric identity

$$0 \leq \chi_0(\mathbf{n}) \left(1 + \operatorname{Re}\{\chi_1(\mathbf{n})(\mathbf{N}\mathbf{n})^{i\gamma'}\}\right) \left(J + \operatorname{Re}\{\chi_1(\mathbf{n})(\mathbf{N}\mathbf{n})^{i\gamma_1}\}\right)^2.$$

In the usual way, it follows that

$$(9.2) \quad \begin{aligned} 0 \leq & (J^2 + \frac{1}{2})\{\mathcal{P}(\sigma, \chi_0) + \mathcal{P}(\sigma + i\gamma', \chi_1)\} + \\ & + J\mathcal{P}(\sigma + i(\gamma_1 + \gamma'), \chi_1^2) + 2J\mathcal{P}(\sigma + i\gamma_1, \chi_1) + J\mathcal{P}(\sigma + i(\gamma_1 - \gamma'), \chi_0) \\ & + \frac{1}{4}\mathcal{P}(\sigma + i(2\gamma_1 + \gamma'), \chi_1^3) + \frac{1}{2}\mathcal{P}(\sigma + 2i\gamma_1, \chi_1^2) + \frac{1}{4}\mathcal{P}(\sigma + i(2\gamma_1 - \gamma'), \chi_1) \end{aligned}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$. To each term $\mathcal{P}(\cdot, \chi_1^r)$ above, we apply Lemma 5.4 extracting zeros depending on the order of χ_1 and the value of r . We divide our argument into cases.

(i) ($\operatorname{ord} \chi_1 \geq 4$) Extract $\{\rho_1, \rho'\}$ from $\mathcal{P}(\cdot, \chi_1^r)$ when $r = 1$. From (9.2), we deduce

$$(9.3) \quad 0 \leq (J^2 + \frac{1}{2})P_4(1) - (J^2 + \frac{1}{2})P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) - 2JP_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) - A + \lambda\psi + \epsilon$$

where $\psi = (J^2 + 3J + \frac{3}{2})\phi\frac{\mathcal{L}\chi_1}{\mathcal{L}} + (J^2 + J + \frac{1}{2})\phi\frac{\mathcal{L}_0}{\mathcal{L}}$, and

$$A = \operatorname{Re}\left\{(J^2 + \frac{3}{4})P_4\left(\frac{\lambda}{\lambda + \lambda_1 + it_1}\right) + 2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda' + it_1}\right) - J \cdot P_4\left(\frac{\lambda}{\lambda + it_1}\right)\right\}$$

with $t_1 = \mu' - \mu_1$. One can easily verify that $J^2 + 3J + \frac{3}{2} \leq 3 \cdot (J^2 + J + \frac{1}{2})$ and so by Lemma 3.1, we may more simply take $\psi = 2(J+1)^2\phi$ in (9.3). By Lemma 5.6, assumption (9.1) implies $A \geq 0$, completing the proof of case (i).

(ii) ($\operatorname{ord} \chi_1 = 3$) Extract $\{\rho_1, \rho'\}$ or $\{\overline{\rho_1}, \overline{\rho'}\}$ from $\mathcal{P}(\cdot, \chi_1^r)$ when $r = 1$ or 2 respectively. Then by (9.2),

$$(9.4) \quad 0 \leq (J^2 + \frac{1}{2})P_4(1) - (J^2 + \frac{1}{2})P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) - 2JP_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) - A - B + \lambda\psi + \epsilon$$

where $\psi = (J^2 + 3J + \frac{5}{4})\phi\frac{\mathcal{L}\chi_1}{\mathcal{L}} + (J^2 + J + \frac{3}{4})\phi\frac{\mathcal{L}_0}{\mathcal{L}}$, the quantity A is as defined in case (i), and

$$B = \operatorname{Re}\left\{J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + it_2}\right) + \frac{1}{2} \cdot P_4\left(\frac{\lambda}{\lambda + \lambda' + it_2}\right) - \frac{1}{4} \cdot P_4\left(\frac{\lambda}{\lambda + it_2}\right)\right\}$$

with $t_2 = \mu' + 2\mu_1$. Again, one can check that $J^2 + 3J + \frac{5}{4} \leq 3 \cdot (J^2 + J + \frac{3}{4})$ and so by Lemma 3.1, we may take $\psi = 2(J+1)^2\phi$ in (9.4). Similar to (i), Lemma 5.6 and assumption (9.1) imply $A, B \geq 0$.

(iii) ($\text{ord } \chi_1 = 2$) *Extract* $\{\rho_1, \overline{\rho_1}, \rho'\}$ from $\mathcal{P}(\cdot, \chi_1^r)$ when $r = 1$ or 3 . Again, apply Lemma 5.4 to the terms in (9.2) except with a slightly more careful analysis. We outline these modifications here.

- Write $2J \cdot \mathcal{P}(\sigma + i\gamma_1, \chi_1) = J \cdot \mathcal{P}(\sigma + i\gamma_1, \chi_1) + J \cdot \mathcal{P}(\sigma + i\gamma_1, \chi_1)$. Extract $\{\rho_1, \overline{\rho_1}, \rho'\}$ from the first term and extract $\{\rho_1, \overline{\rho_1}, \overline{\rho'}\}$ from the second term.
- For $\frac{1}{4}\mathcal{P}(\sigma + i(2\gamma_1 + \gamma'), \chi_1)$ and $\frac{1}{4}\mathcal{P}(\sigma + i(2\gamma_1 - \gamma'), \chi_1)$, extract $\{\rho_1, \rho'\}$ and $\{\rho_1, \overline{\rho'}\}$ respectively.

With these modifications, (9.2) overall yields

$$\begin{aligned}
(9.5) \quad 0 \leq & (J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2})P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) - 2JP_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) + \lambda(\psi + \epsilon) \\
& - \text{Re}\left\{(J^2 + \tfrac{3}{4})P_4\left(\frac{\lambda}{\lambda + \lambda_1 + it_1}\right) + J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda' + it_1}\right) - J \cdot P_4\left(\frac{\lambda}{\lambda + it_1}\right)\right\} \\
& - \text{Re}\left\{(J^2 + \tfrac{3}{4}) \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + it_3}\right) + J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda' + it_3}\right) - J \cdot P_4\left(\frac{\lambda}{\lambda + it_3}\right)\right\} \\
& - \text{Re}\left\{2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + it_4}\right) + \tfrac{1}{2} \cdot P_4\left(\frac{\lambda}{\lambda + \lambda' + it_4}\right) - \tfrac{1}{2} \cdot P_4\left(\frac{\lambda}{\lambda + it_4}\right)\right\}
\end{aligned}$$

where $t_1 = \mu' - \mu_1$; $t_3 = \mu' + \mu_1$; $t_4 = 2\mu_1$; and $\psi = (J^2 + 2J + 1)\phi \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} + (J^2 + 2J + 1)\phi \frac{\mathcal{L}_0}{\mathcal{L}}$. Trivially $J^2 + 2J + 1 \leq 3 \cdot (J^2 + 2J + 1)$ and so by Lemma 3.1, we may more simply take $\psi = 2(J + 1)^2\phi$. The three terms $\text{Re}\{\dots\}$ in (9.5) are all ≥ 0 by Lemma 5.6 and (9.1) and hence can be ignored.

This completes the proof in all cases. \square

A suitable numerical version of Lemma 9.1 produces Table 8.

9.1.2. χ_1 principal.

Lemma 9.2. *Assume χ_1 is principal, ρ_1 is complex, and ρ' is real. Suppose f satisfies Conditions 1 and 2. For $\epsilon > 0$, provided \mathcal{L} is sufficiently large depending on ϵ and f , the following holds:*

$$0 \leq 2F(-\lambda') - 2F(\lambda_1 - \lambda') - F(0) + 2\phi f(0) + \epsilon.$$

Proof. This is analogous to Lemma 8.1. To be brief, use (8.2) with $(\chi, \gamma) = (\chi_0, \gamma_1)$ and $\sigma = \beta'$ and apply Lemma 6.3 extracting $\{\rho', \rho_1, \overline{\rho_1}\}$ from $\mathcal{K}(\beta', \chi_0)$ and $\{\rho', \rho_1\}$ from $\mathcal{K}(\beta' + i\gamma_1, \chi_0)$. \square

A numerical version of Lemma 9.2 yields bounds for λ' with $f = f_\lambda$ taken from [HB95, Lemma 7.2], producing Table 9. The remaining case consists of χ_1 principal with both ρ_1 and ρ' complex.

Lemma 9.3. *Assume χ_1 is principal, ρ_1 is complex and ρ' is complex. Let $\lambda > 0$ and $J > 0$. If \mathcal{L} is sufficiently large depending on ϵ, λ and J then*

$$0 \leq (J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2}) \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) - 2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) + 2(J + 1)^2\phi\lambda + \epsilon$$

provided both of the following hold:

$$(9.6) \quad \frac{1}{(\lambda + \lambda_1)^4} + \frac{J_0}{(\lambda + \lambda')^4} > \frac{1}{\lambda^4} \quad \text{and} \quad \frac{2}{(\lambda + \lambda_1)^4} + \frac{J_1}{(\lambda + \lambda')^4} > \frac{1}{\lambda^4},$$

$\lambda_1 \leq$	$\lambda' \geq$	λ	J	$\lambda_1 \leq$	$\lambda' \geq$	λ	J
.1227	.7391	1.097	.7788	.210	.4353	1.264	.8073
.125	.7266	1.104	.7797	.215	.4241	1.269	.8087
.130	.7007	1.120	.7817	.220	.4132	1.273	.8100
.135	.6766	1.135	.7836	.225	.4027	1.276	.8114
.140	.6540	1.149	.7854	.230	.3926	1.280	.8127
.145	.6328	1.162	.7872	.235	.3828	1.283	.8140
.150	.6128	1.174	.7889	.240	.3733	1.285	.8153
.155	.5939	1.185	.7906	.245	.3641	1.288	.8166
.160	.5759	1.195	.7923	.250	.3552	1.290	.8179
.165	.5589	1.204	.7939	.255	.3465	1.292	.8191
.170	.5427	1.213	.7955	.260	.3381	1.294	.8204
.175	.5272	1.221	.7971	.265	.3300	1.295	.8216
.180	.5124	1.229	.7986	.270	.3220	1.296	.8229
.185	.4982	1.236	.8001	.275	.3143	1.297	.8241
.190	.4846	1.242	.8016	.280	.3068	1.298	.8253
.195	.4715	1.249	.8030	.285	.2995	1.299	.8265
.200	.4590	1.254	.8045	.290	.2924	1.299	.8277
.205	.4469	1.259	.8059	.2909	.2911	1.299	.8279

TABLE 8. Bounds for λ' with χ_1 or ρ_1 complex and χ_1 non-principal

$\lambda_1 \leq$	$\lambda' \geq$	λ	$\lambda_1 \leq$	$\lambda' \geq$	λ
.0875	1.836	1.189	.22	.7994	1.023
.09	1.803	1.185	.23	.7522	1.013
.10	1.681	1.170	.24	.7073	1.002
.11	1.572	1.156	.25	.6646	.9917
.12	1.472	1.142	.26	.6239	.9813
.13	1.381	1.129	.27	.5851	.9711
.14	1.297	1.116	.28	.5480	.9609
.15	1.220	1.103	.29	.5126	.9508
.16	1.148	1.091	.30	.4787	.9407
.17	1.080	1.079	.31	.4462	.9307
.18	1.017	1.068	.32	.4150	.9208
.19	.9578	1.056	.33	.3851	.9108
.20	.9020	1.045	.34	.3565	.9009
.21	.8493	1.034	.3443	.3445	.8966

TABLE 9. Bounds for λ' with χ_1 principal, ρ_1 complex and ρ' real

where $J_0 = \min\{J + \frac{3}{4J}, 4J\}$ and $J_1 = 4J/(J^2 + 1)$.

Proof. Analogous to Lemma 9.1 but we exchange the roles of ρ_1 and ρ' using the trigonometric identity

$$0 \leq \chi_0(\mathbf{n}) \left(1 + \operatorname{Re}\{(\mathbf{N}\mathbf{n})^{i\gamma_1}\}\right) \left(J + \operatorname{Re}\{(\mathbf{N}\mathbf{n})^{i\gamma'}\}\right)^2.$$

Writing $\mathcal{P}(s) = \mathcal{P}(s, \chi_0; P_4)$, it follows in the usual way that

$$(9.7) \quad 0 \leq (J^2 + \tfrac{1}{2})\{\mathcal{P}(\sigma) + \mathcal{P}(\sigma + i\gamma_1)\} + J\mathcal{P}(\sigma + i(\gamma' + \gamma_1)) + 2J\mathcal{P}(\sigma + i\gamma') + J\mathcal{P}(\sigma + i(\gamma' - \gamma_1)) \\ + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma' + \gamma_1)) + \tfrac{1}{2}\mathcal{P}(\sigma + 2i\gamma') + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma' - \gamma_1))$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$. Next, apply Lemma 5.4 to each term according to the following outline:

- $\mathcal{P}(\sigma)$ and $\mathcal{P}(\sigma + i\gamma')$ extract all 4 zeros $\{\rho_1, \overline{\rho_1}, \rho', \overline{\rho'}\}$.
- $\mathcal{P}(\sigma + i\gamma_1)$ and $\mathcal{P}(\sigma + i(\gamma' + \gamma_1))$ extract only $\{\rho_1, \rho', \overline{\rho'}\}$.
- $\mathcal{P}(\sigma + i(\gamma' - \gamma_1))$ extract only $\{\overline{\rho_1}, \rho', \overline{\rho'}\}$.
- $\mathcal{P}(\sigma + i(2\gamma' + \gamma_1))$ and $\mathcal{P}(\sigma + i(2\gamma' - \gamma_1))$ extract $\{\rho_1, \rho'\}$ and $\{\overline{\rho_1}, \rho'\}$ respectively.
- $\mathcal{P}(\sigma + 2i\gamma')$ extract only $\{\rho_1, \overline{\rho_1}, \rho'\}$.

When necessary, we utilize that $P_4(\overline{X}) = \overline{P_4(X)}$. Then overall we obtain:

$$(9.8) \quad 0 \leq (J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2})P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) - 2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda'}\right) + 2\phi(J+1)^2\lambda + \epsilon \\ - \sum_{r=1}^7 \operatorname{Re}\left\{A_r \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + it_r}\right) + B_r \cdot P_4\left(\frac{\lambda}{\lambda + \lambda' + it_r}\right) - C_r \cdot P_4\left(\frac{\lambda}{\lambda + it_r}\right)\right\}$$

where

r	1	2	3	4	5	6	7
t_r	μ_1	μ'	$\mu' + \mu_1$	$\mu' - \mu_1$	$2\mu'$	$2\mu' + \mu_1$	$2\mu' - \mu_1$
A_r	$2J^2 + 1$	$2J$	$2J$	$2J$	$1/2$	$1/2$	$1/2$
B_r	$2J$	$2J^2 + 3/2$	$J^2 + 3/4$	$J^2 + 3/4$	$2J$	J	J
C_r	$J^2 + 1/2$	$2J$	J	J	$1/2$	$1/4$	$1/4$

It suffices to show the sum over r in (9.8) is non-negative. By Lemma 5.6, the sum is ≥ 0 provided

$$\frac{A_r}{(\lambda + \lambda_1)^4} + \frac{B_r}{(\lambda + \lambda')^4} > \frac{C_r}{\lambda^4} \quad \text{for } r = 1, 2, \dots, 7.$$

After inspection, the most stringent conditions are $r = 1, 2$ and 5 , which are implied by assumption (9.6). \square

This produces Table 10 in the usual fashion.

9.1.3. *Summary of bounds.* We collect the results in the subsection into a single proposition for the reader's convenience.

Proposition 9.4. *Assume χ_1 or ρ_1 is complex. Provided \mathcal{L} is sufficiently large, we have the following:*

- If χ_1 is non-principal then $\lambda' \geq 0.2909$ and the bounds for λ' in Table 8 apply.*
- If χ_1 is principal then $\lambda' \geq 0.2909$ and the bounds for λ' in Table 10 apply.*

Proof. If χ_1 is non-principal, then the only bounds available come from Table 8. If χ_1 is principal, then upon comparing Tables 9 and 10, one finds that the latter gives weaker bounds. \square

$\lambda_1 \leq$	$\lambda' \geq$	λ	J	$\lambda_1 \leq$	$\lambda' \geq$	λ	J
.0875	.5330	1.155	.8815	.195	.3749	1.280	.8472
.090	.5278	1.161	.8804	.200	.3696	1.282	.8460
.095	.5179	1.171	.8782	.205	.3645	1.284	.8449
.100	.5083	1.181	.8762	.210	.3594	1.286	.8437
.105	.4991	1.190	.8742	.215	.3545	1.288	.8426
.110	.4902	1.198	.8723	.220	.3497	1.290	.8415
.115	.4817	1.206	.8704	.225	.3449	1.291	.8405
.120	.4734	1.213	.8686	.230	.3403	1.293	.8394
.125	.4654	1.220	.8669	.235	.3358	1.294	.8384
.130	.4577	1.226	.8652	.240	.3314	1.295	.8374
.135	.4502	1.232	.8636	.245	.3270	1.296	.8364
.140	.4429	1.238	.8620	.250	.3228	1.297	.8354
.145	.4359	1.243	.8605	.255	.3186	1.297	.8344
.150	.4290	1.248	.8590	.260	.3145	1.298	.8335
.155	.4223	1.252	.8576	.265	.3106	1.298	.8326
.160	.4159	1.257	.8562	.270	.3066	1.299	.8317
.165	.4096	1.261	.8548	.275	.3028	1.299	.8308
.170	.4034	1.265	.8534	.280	.2990	1.299	.8299
.175	.3974	1.268	.8521	.285	.2953	1.299	.8290
.180	.3916	1.271	.8509	.290	.2917	1.299	.8281
.185	.3859	1.274	.8496	.2909	.2911	1.299	.8280
.190	.3804	1.277	.8484				

TABLE 10. Bounds for λ' with χ_1 principal, ρ_1 complex and ρ' complex

9.2. **Bounds for λ_2 .** Before dividing into cases, we begin with the following lemma analogous to Lemma 9.1.

Lemma 9.5. *Assume χ_1 or ρ_1 is complex. Suppose f satisfies Conditions 1 and 2. For $\epsilon > 0$, provided \mathcal{L} is sufficiently large depending on ϵ and f , the following holds:*

(a) *If χ_1, χ_2 are non-principal, then*

$$0 \leq F(-\lambda_1) - F(0) - F(\lambda_2 - \lambda_1) + 4\phi f(0) + \epsilon.$$

(b) *If χ_1 is principal, then ρ_1 is complex, χ_2 is non-principal and*

$$0 \leq F(-\lambda_1) - F(0) - F(\lambda_2 - \lambda_1) + \operatorname{Re}\{F(-\lambda_1 + i\mu_1) - F(i\mu_1) - F(\lambda_2 - \lambda_1 + i\mu_1)\} + 4\phi f(0) + \epsilon.$$

(c) *If χ_2 is principal, then χ_1 is non-principal and*

$$0 \leq F(-\lambda_1) - F(0) - F(\lambda_2 - \lambda_1) + \operatorname{Re}\{F(-\lambda_1 + i\mu_2) - F(i\mu_2) - F(\lambda_2 - \lambda_1 + i\mu_2)\} + 4\phi f(0) + \epsilon.$$

Proof. The arguments involved are very similar to Lemma 8.1 and Lemma 8.8 so we omit most of the details. Briefly, use (8.1) by setting $(\chi, \rho) = (\chi_1, \rho_1)$ and $(\chi_*, \rho_*) = (\chi_2, \rho_2)$ and $\sigma = \beta_1$, which gives

$$(9.9) \quad \begin{aligned} 0 \leq & \mathcal{K}(\beta_1, \chi_0) + \mathcal{K}(\beta_1 + i\gamma_1, \chi_1) + \mathcal{K}(\beta_1 + i\gamma_2, \chi_2) \\ & + \frac{1}{2}\mathcal{K}(\beta_1 + i(\gamma_1 + \gamma_2), \chi_1\chi_2) + \frac{1}{2}\mathcal{K}(\beta_1 + i(\gamma_1 - \gamma_2), \chi_1\overline{\chi_2}). \end{aligned}$$

$\lambda_1 \leq$	$\lambda_2 \geq$	λ	$\lambda_1 \leq$	$\lambda_2 \geq$	λ
.1227	.4890	.3837	.220	.3715	.4380
.13	.4779	.3888	.225	.3668	.4402
.135	.4706	.3922	.230	.3622	.4423
.140	.4635	.3955	.235	.3576	.4444
.145	.4566	.3986	.240	.3532	.4465
.150	.4499	.4017	.245	.3488	.4486
.155	.4433	.4047	.250	.3446	.4506
.160	.4370	.4077	.255	.3404	.4526
.165	.4308	.4105	.260	.3363	.4545
.170	.4247	.4133	.265	.3322	.4564
.175	.4188	.4160	.270	.3283	.4583
.180	.4131	.4187	.275	.3244	.4602
.185	.4075	.4213	.280	.3205	.4620
.190	.4020	.4238	.285	.3168	.4638
.195	.3966	.4263	.290	.3131	.4656
.200	.3914	.4287	.295	.3094	.4673
.205	.3862	.4311	.300	.3059	.4690
.210	.3812	.4334	.3034	.3035	.4702
.215	.3763	.4357			

TABLE 11. Bounds for λ_2 with χ_1 or ρ_1 complex and χ_1, χ_2 non-principal

Apply Lemma 6.3 to each $\mathcal{K}(*, *)$ term, extracting zeros ρ_1 or ρ_2 whenever possible, depending on the cases. Recall $\chi_1\chi_2$ and $\chi_1\overline{\chi_2}$ are always non-principal by construction (see Section 3). \square

9.2.1. χ_1 and χ_2 non-principal. A numerical version of Lemma 9.5 suffices here.

Lemma 9.6. *Assume χ_1 or ρ_1 is complex with χ_1, χ_2 non-principal. Let $\epsilon > 0$ and for $b > 0$, assume $0 < \lambda_1 \leq b$. Suppose, for some $\tilde{\lambda}_b > 0$, we have*

$$F(-b) - F(0) - F(\tilde{\lambda}_b - b) + 4\phi f(0) \leq 0$$

Then $\lambda_2 \geq \tilde{\lambda}_b - \epsilon$ provided \mathcal{L} is sufficiently large depending on ϵ and f .

Proof. Analogous to Lemma 8.10 using Lemma 9.5 in place of Lemma 8.8. Hence, we omit the proof. \square

This produces Table 11 by taking $f = f_\lambda$ from [HB95, Lemma 7.2] with parameter $\lambda = \lambda(b)$.

9.2.2. χ_1 principal or χ_2 is principal. When χ_2 is principal and ρ_2 is real, a numerical version of Lemma 9.5 suffices.

Lemma 9.7. *Assume χ_1 or ρ_1 is complex. Further assume χ_2 is principal and ρ_2 is real. Let $\epsilon > 0$ and for $b > 0$, assume $0 < \lambda_1 \leq b$. Suppose, for some $\tilde{\lambda}_b > 0$, we have*

$$F(-b) - F(0) - F(\tilde{\lambda}_b - b) + 2\phi f(0) \leq 0$$

Then $\lambda_2 \geq \tilde{\lambda}_b - \epsilon$ provided \mathcal{L} is sufficiently large depending on ϵ and f .

$\lambda_1 \leq$	$\lambda_2 \geq$	λ	$\lambda_1 \leq$	$\lambda_2 \geq$	λ
.1227	1.221	.6530	.37	.8149	.8425
.13	1.203	.6620	.39	.7932	.8526
.15	1.155	.6846	.41	.7725	.8622
.17	1.112	.7049	.43	.7526	.8714
.19	1.073	.7234	.45	.7335	.8803
.21	1.037	.7403	.47	.7152	.8889
.23	1.003	.7560	.49	.6977	.8971
.25	.9710	.7707	.51	.6807	.9051
.27	.9412	.7844	.53	.6644	.9128
.29	.9132	.7973	.55	.6487	.9203
.31	.8867	.8095	.57	.6336	.9276
.33	.8615	.8210	.59	.6189	.9346
.35	.8377	.8320	.6068	.6070	.9404

TABLE 12. Bounds for λ_2 with χ_1 or ρ_1 complex and χ_2 principal and ρ_2 real

This produces Table 12 by taking f from [HB95, Lemma 7.2] with parameter $\lambda = \lambda(b)$. Now, when χ_1 is principal or when χ_2 is principal and ρ_2 is complex, we employ the “polynomial method”.

Lemma 9.8. *Suppose χ_j is principal and ρ_j is complex, and let $\chi_k \neq \chi_j$. Let $\epsilon, \lambda, J > 0$. If \mathcal{L} is sufficiently large depending on ϵ, λ and J , then*

$$0 \leq (J^2 + \tfrac{1}{2})\{P_4(1) - P_4(\tfrac{\lambda}{\lambda + \lambda_k})\} - 2JP_4(\tfrac{\lambda}{\lambda + \lambda_j}) + 2\phi(J+1)^2\lambda + \epsilon$$

provided

$$(9.10) \quad \frac{J_0}{(\lambda + \lambda_j)^4} + \frac{1}{(\lambda + \lambda_k)^4} > \frac{1}{\lambda^4} \quad \text{with } J_0 = \min\{J + \tfrac{3}{4J}, 4J\}.$$

Proof. Write $\mathcal{P}(s, \chi) = \mathcal{P}(s, \chi; P_4)$. We begin with the inequality

$$0 \leq \chi_0(\mathbf{n})(1 + \operatorname{Re}\{\chi_k(\mathbf{n})(\mathbf{Nn})^{-i\gamma_k}\})(J + \operatorname{Re}\{(\mathbf{Nn})^{-i\gamma_j}\})^2$$

It follows in the usual fashion that

$$(9.11) \quad \begin{aligned} 0 \leq & (J^2 + \tfrac{1}{2})\{\mathcal{P}(\sigma, \chi_0) + \mathcal{P}(\sigma + i\gamma_k, \chi_k)\} + \\ & + J\mathcal{P}(\sigma + i(\gamma_j + \gamma_k), \chi_k) + 2J\mathcal{P}(\sigma + i\gamma_j, \chi_0) + J\mathcal{P}(\sigma + i(\gamma_j - \gamma_k), \overline{\chi_k}) \\ & + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma_j + \gamma_k), \chi_k) + \tfrac{1}{2}\mathcal{P}(\sigma + 2i\gamma_j, \chi_0) + \tfrac{1}{4}\mathcal{P}(\sigma + i(2\gamma_j - \gamma_k), \overline{\chi_k}) \end{aligned}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$. Next, apply Lemma 5.4 to each $\mathcal{P}(*, *)$ term in (9.11) extracting $\{\rho_j, \overline{\rho_j}\}$ from χ_0 -terms, ρ_k from the χ_k -terms, and $\overline{\rho_k}$ from $\overline{\chi_k}$ -terms. When necessary, we also use that $P_4(\overline{X}) = \overline{P_4(X)}$. Then overall

$$(9.12) \quad 0 \leq (J^2 + \tfrac{1}{2})P_4(1) - (J^2 + \tfrac{1}{2})P_4(\tfrac{\lambda}{\lambda + \lambda_k}) - 2JP_4(\tfrac{\lambda}{\lambda + \lambda_j}) + \psi\lambda + \epsilon - A - B$$

where

$$A = \operatorname{Re}\left\{(2J^2 + \frac{3}{2})P_4\left(\frac{\lambda}{\lambda + \lambda_j + i\mu_j}\right) + 2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_k + i\mu_j}\right) - 2J \cdot P_4\left(\frac{\lambda}{\lambda + i\mu_j}\right)\right\},$$

$$B = \operatorname{Re}\left\{2J \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_j + 2i\mu_j}\right) + \frac{1}{2} \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_k + 2i\mu_j}\right) - \frac{1}{2} \cdot P_4\left(\frac{\lambda}{\lambda + 2i\mu_j}\right)\right\},$$

and $\psi = (J^2 + 2J + 1)\phi \frac{\mathcal{L}_{\chi_k}}{\mathcal{L}} + (J^2 + 2J + 1)\phi \frac{\mathcal{L}_0}{\mathcal{L}}$. Trivially $J^2 + 2J + 1 \leq 3 \cdot (J^2 + 2J + 1)$ and so by Lemma 3.1, we may more simply take $\psi = 2(J + 1)^2\phi$ in (9.12). From Lemma 5.6 and (9.10), it follows $A, B \geq 0$. \square

We record a numerical version of Lemma 9.8 without proof.

Corollary 9.9. *Suppose χ_1 or ρ_1 is complex. For $b > 0$, assume $0 < \lambda_1 \leq b$ and let $\lambda, J > 0$. Denote $J_0 := \min\{J + \frac{3}{4J}, 4J\}$. Assume one of the following holds:*

(a) χ_1 is principal, ρ_1 is complex. Further there exists $\tilde{\lambda}_b \in [0, \infty)$ satisfying

$$0 = (J^2 + \frac{1}{2})(P_4(1) - P_4(\frac{\lambda}{\lambda + \tilde{\lambda}_b})) - 2J \cdot P_4(\frac{\lambda}{\lambda + b}) + 2\phi(J + 1)^2\lambda + \epsilon.$$

and

$$\frac{J_0}{(\lambda + b)^4} + \frac{1}{(\lambda + \tilde{\lambda}_b)^4} > \frac{1}{\lambda^4}.$$

(b) χ_2 is principal, ρ_2 is complex. Further there exists $\tilde{\lambda}_b \in [0, \infty)$ satisfying

$$0 = (J^2 + \frac{1}{2})(P_4(1) - P_4(\frac{\lambda}{\lambda + \tilde{\lambda}_b})) - 2J \cdot P_4(\frac{\lambda}{\lambda + \tilde{\lambda}_b}) + 2\phi(J + 1)^2\lambda + \epsilon$$

and

$$\frac{1}{(\lambda + b)^4} + \frac{J_0}{(\lambda + \tilde{\lambda}_b)^4} > \frac{1}{\lambda^4}.$$

Then, in either case, $\lambda_2 \geq \tilde{\lambda}_b - \epsilon$ for \mathcal{L} sufficiently large depending on ϵ, b, λ and J .

This produces Tables 13 and 14.

9.2.3. Summary of bounds. We collect the results in the subsection into a single proposition for the reader's convenience.

Proposition 9.10. *Assume χ_1 or ρ_1 is complex. Provided \mathcal{L} is sufficiently large, the following holds:*

- (a) *If χ_1 is non-principal, then $\lambda_2 \geq 0.2909$ and the bounds for λ_2 in Table 14 apply.*
- (b) *If χ_1 is principal, then $\lambda_2 \geq 0.2909$ and the bounds for λ_2 in Table 13 apply.*

Proof. If χ_1 is non-principal then one compares Table 11, Table 12 and Table 14 and finds that the last one gives the weakest bounds. If χ_1 is principal, then the only bounds available come from Table 13. \square

10. ZERO-FREE REGION

Proof of Theorem 1.1: If χ_1 and ρ_1 are both real, then Theorem 1.1 is implied by Propositions 8.7 and 8.13. Thus, it remains to consider when χ_1 or ρ_1 is complex, dividing our cases according to the order of χ_1 .

$\lambda_1 \leq$	$\lambda_2 \geq$	λ	J	$\lambda_1 \leq$	$\lambda_2 \geq$	λ	J
.0875	1.017	.9321	.7627	.195	.4715	1.249	.8030
.090	.9892	.9474	.7640	.200	.4590	1.254	.8045
.095	.9385	.9760	.7666	.205	.4469	1.259	.8059
.100	.8937	1.002	.7690	.210	.4353	1.264	.8073
.105	.8537	1.026	.7713	.215	.4241	1.269	.8087
.110	.8175	1.048	.7735	.220	.4132	1.273	.8100
.115	.7846	1.069	.7757	.225	.4027	1.276	.8114
.120	.7544	1.087	.7777	.230	.3926	1.280	.8127
.125	.7266	1.104	.7797	.235	.3828	1.283	.8140
.130	.7007	1.120	.7817	.240	.3733	1.285	.8153
.135	.6766	1.135	.7836	.245	.3641	1.288	.8166
.140	.6540	1.149	.7854	.250	.3552	1.290	.8179
.145	.6328	1.162	.7872	.255	.3465	1.292	.8191
.150	.6128	1.174	.7889	.260	.3381	1.294	.8204
.155	.5939	1.185	.7906	.265	.3300	1.295	.8216
.160	.5759	1.195	.7923	.270	.3220	1.296	.8229
.165	.5589	1.204	.7939	.275	.3143	1.297	.8241
.170	.5427	1.213	.7955	.280	.3068	1.298	.8253
.175	.5272	1.221	.7971	.285	.2995	1.299	.8265
.180	.5124	1.229	.7986	.290	.2924	1.299	.8277
.185	.4982	1.236	.8001	.2909	.2911	1.299	.8279
.190	.4846	1.242	.8016				

TABLE 13. Bounds for λ_2 with χ_1 principal and ρ_1 complex

χ_1 **has order** ≥ 5 . We begin with the inequality

$$(10.1) \quad 0 \leq \chi_0(\mathbf{n}) \left(3 + 10 \cdot \operatorname{Re}\{\chi_1(\mathbf{n})(\mathbf{N}\mathbf{n})^{-i\gamma_1}\} \right)^2 \left(9 + 10 \cdot \operatorname{Re}\{\chi_1(\mathbf{n})(\mathbf{N}\mathbf{n})^{-i\gamma_1}\} \right)^2$$

which was also used in [HB95, Section 9]. This will also be roughly optimal for our purposes. We shall use the smoothed explicit inequality with a weight f satisfying Conditions 1 and 2. By the usual arguments, we expand out the above identity, multiply by the appropriate factor and sum over \mathbf{n} . Overall this yields

$$(10.2) \quad \begin{aligned} 0 \leq & 14379 \cdot \mathcal{K}(\sigma, \chi_0) + 24480 \cdot \mathcal{K}(\sigma + i\gamma_1, \chi_1) + 14900 \cdot \mathcal{K}(\sigma + 2i\gamma_1, \chi_1^2) \\ & + 6000 \cdot \mathcal{K}(\sigma + 3i\gamma_1, \chi_1^3) + 1250 \cdot \mathcal{K}(\sigma + 4i\gamma_1, \chi_1^4) \end{aligned}$$

where $\mathcal{K}(s, \chi) = \mathcal{K}(s, \chi; f)$ and $\sigma = 1 - \frac{\lambda^*}{\varepsilon}$ with constant λ^* satisfying

$$\lambda_1 \leq \lambda^* \leq \min\{\lambda', \lambda_2\}.$$

Now, apply Lemma 6.3 to each term in (10.2) and consider cases depending on $\operatorname{ord} \chi_1$. For $\mathcal{K}(\sigma + ni\gamma_1, \chi_1^n)$:

- ($\operatorname{ord} \chi_1 \geq 6$) Extract $\{\rho_1\}$ if $n = 1$ only.
- ($\operatorname{ord} \chi_1 = 5$) Set $\lambda^* = \lambda_1$ and extract $\{\rho_1\}$ if $n = 1$ only.

It follows that

$$(10.3) \quad 0 \leq 14379 \cdot F(-\lambda^*) - 24480 \cdot F(\lambda_1 - \lambda^*) + Bf(0)\phi + \epsilon$$

$\lambda_1 \leq$	$\lambda_2 \geq$	λ	J	$\lambda_1 \leq$	$\lambda_2 \geq$	λ	J
.1227	.4691	1.217	.8677	.215	.3545	1.288	.8426
.125	.4654	1.220	.8669	.220	.3497	1.290	.8415
.130	.4577	1.226	.8652	.225	.3449	1.291	.8405
.135	.4502	1.232	.8636	.230	.3403	1.293	.8394
.140	.4429	1.238	.8620	.235	.3358	1.294	.8384
.145	.4359	1.243	.8605	.240	.3314	1.295	.8374
.150	.4290	1.248	.8590	.245	.3270	1.296	.8364
.155	.4223	1.252	.8576	.250	.3228	1.297	.8354
.160	.4159	1.257	.8562	.255	.3186	1.297	.8344
.165	.4096	1.261	.8548	.260	.3145	1.298	.8335
.170	.4034	1.265	.8534	.265	.3106	1.298	.8326
.175	.3974	1.268	.8521	.270	.3066	1.299	.8317
.180	.3916	1.271	.8509	.275	.3028	1.299	.8308
.185	.3859	1.274	.8496	.280	.2990	1.299	.8299
.190	.3804	1.277	.8484	.285	.2953	1.299	.8290
.195	.3749	1.280	.8472	.290	.2917	1.299	.8281
.200	.3696	1.282	.8460	.2909	.2911	1.299	.8280
.205	.3645	1.284	.8449				
.210	.3594	1.286	.8437				

TABLE 14. Bounds for λ_2 with χ_1 or ρ_1 complex and χ_2 principal and ρ_2 complex

where $B = 14379 \cdot \frac{\mathcal{L}_0}{\mathcal{L}} + 46630 \cdot \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}}$. From Lemma 3.1, $B \leq 57516 + 3493 \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} \leq 62174$ so (10.3) reduces to

$$(10.4) \quad 0 \leq 14379 \cdot F(-\lambda^*) - 24480 \cdot F(\lambda_1 - \lambda^*) + 62174 \phi f(0) + \epsilon.$$

We now consider cases.

- ($\text{ord } \chi_1 \geq 6$) Without loss, we may assume $\lambda_1 \leq 0.180$. From Propositions 9.4 and 9.10, we may take $\lambda^* = 0.3916$. Choose f according to [HB95, Lemma 7.1] with parameters $\theta = 1$ and $\lambda = 0.243$. Then (10.4) implies $\lambda_1 \geq 0.1764$.
- ($\text{ord } \chi_1 = 5$) Since $\lambda^* = \lambda_1$ in this case, (10.4) becomes

$$0 \leq 14379 \cdot F(-\lambda_1) - 24480 \cdot F(0) + 62174 f(0) \phi + \epsilon.$$

We choose f according to [HB95, Lemma 7.5] with $k = 24480/14379$ giving $\theta = 1.1580\dots$ and

$$\lambda_1^{-1} \cos^2 \theta \leq \frac{1}{4} \cdot \frac{62174}{14379} + \epsilon$$

whence $\lambda_1 \geq 0.1489$.

Remark. To bound the quantity B for $\text{ord } \chi_1 \geq 5$, the proof above uses that $\vartheta \geq \frac{3}{4}$ leading to some minor loss in the lower bound for λ_1 . If one uses $\vartheta = 1$, say, then one can slightly improve this lower bound.

χ_1 **has order 2, 3 or 4.** We use the same identity (10.1) but instead will apply the “polynomial method” with $P_4(X)$. In the usual way, it follows from (10.1) that

$$(10.5) \quad \begin{aligned} 0 \leq & 14379 \cdot \mathcal{P}(\sigma, \chi_0) + 24480 \cdot \mathcal{P}(\sigma + i\gamma_1, \chi_1) + 14900 \cdot \mathcal{P}(\sigma + 2i\gamma_1, \chi_1^2) \\ & + 6000 \cdot \mathcal{P}(\sigma + 3i\gamma_1, \chi_1^3) + 1250 \cdot \mathcal{P}(\sigma + 4i\gamma_1, \chi_1^4) \end{aligned}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$ with $\lambda > 0$. The above identity will be roughly optimal for our purposes. Now, we apply Lemma 5.4 to each term above and consider cases depending on $\text{ord } \chi_1$. For each term $\mathcal{P}(\sigma + ni\gamma_1, \chi_1^n)$:

- ($\text{ord } \chi_1 = 4$) Extract $\{\rho_1\}$ if $n = 1$ and $\{\overline{\rho_1}\}$ if $n = 3$.
- ($\text{ord } \chi_1 = 3$) Extract $\{\rho_1\}$ if $n = 1$ or 4 and $\{\overline{\rho_1}\}$ if $n = 2$.
- ($\text{ord } \chi_1 = 2$) Extract $\{\rho_1, \overline{\rho_1}\}$ if $n = 1$ or 3 since ρ_1 is necessarily complex.

It then follows that

$$(10.6) \quad 0 \leq 14379 \cdot P_4(1) - 24480 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) + A_{\chi_1} + B_{\chi_1}\phi\lambda + \epsilon$$

with

$$A_{\chi_1} = \begin{cases} \text{Re}\{1250 \cdot P_4\left(\frac{\lambda}{\lambda + 4i\mu_1}\right) - 6000 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 4i\mu_1}\right)\}, & \text{ord } \chi_1 = 4, \\ \text{Re}\{6000 \cdot P_4\left(\frac{\lambda}{\lambda + 3i\mu_1}\right) - 16150 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 3i\mu_1}\right)\}, & \text{ord } \chi_1 = 3, \\ \text{Re}\{14900 \cdot P_4\left(\frac{\lambda}{\lambda + 2i\mu_1}\right) - 30480 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 2i\mu_1}\right)\} \\ \quad + \text{Re}\{1250 \cdot P_4\left(\frac{\lambda}{\lambda + 4i\mu_1}\right) - 6000 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 4i\mu_1}\right)\}, & \text{ord } \chi_1 = 2. \end{cases}$$

and

$$B_{\chi_1} = \begin{cases} 15629 \cdot \frac{\mathcal{L}_0}{\mathcal{L}} + 45380 \cdot \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} & \text{if } \text{ord } \chi_1 = 4, \\ 20379 \cdot \frac{\mathcal{L}_0}{\mathcal{L}} + 40630 \cdot \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} & \text{if } \text{ord } \chi_1 = 3, \\ 30529 \cdot \frac{\mathcal{L}_0}{\mathcal{L}} + 30480 \cdot \frac{\mathcal{L}_{\chi_1}}{\mathcal{L}} & \text{if } \text{ord } \chi_1 = 2. \end{cases}$$

By Lemma 3.1, we observe $B_{\chi_1} \leq 61009$. Furthermore, applying Lemma 5.6 to A_{χ_1} , it follows that $A_{\chi_1} \leq 0$ in all cases provided

$$(10.7) \quad \frac{14900}{\lambda^4} - \frac{30480}{(\lambda + \lambda_1)^4} \leq 0.$$

Thus, (10.6) implies

$$0 \leq 14379 \cdot P_4(1) - 24480 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) + 61009\phi\lambda + \epsilon$$

provided (10.7) holds. Taking $\lambda = 0.9421$ yields $\lambda_1 \geq 0.1227$.

χ_1 is **principal**. Recall in this case we assume ρ_1 is complex. We begin with a slightly different inequality:

$$0 \leq \chi_0(\mathbf{n}) \left(0 + 10 \cdot \operatorname{Re}\{(\mathbf{Nn})^{-i\gamma_1}\} \right)^2 \left(7 + 10 \cdot \operatorname{Re}\{(\mathbf{Nn})^{-i\gamma_1}\} \right)^2.$$

Again using the “polynomial method” with $P_4(X)$, it similarly follows that

$$(10.8) \quad \begin{aligned} 0 \leq & 620 \cdot \mathcal{P}(\sigma, \chi_0) + 1050 \cdot \mathcal{P}(\sigma + i\gamma_1, \chi_0) + 745 \cdot \mathcal{P}(\sigma + 2i\gamma_1, \chi_0) \\ & + 350 \cdot \mathcal{P}(\sigma + 3i\gamma_1, \chi_0) + 125 \cdot \mathcal{P}(\sigma + 4i\gamma_1, \chi_0) \end{aligned}$$

where $\sigma = 1 + \frac{\lambda}{\mathcal{L}}$ with $\lambda > 0$. Apply Lemma 5.4 to each term above, extracting $\{\rho_1, \overline{\rho_1}\}$ since ρ_1 is necessarily complex. It then follows that

$$(10.9) \quad 0 \leq 620 \cdot P_4(1) - 1050 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) + A_0 + 2890\phi\lambda + \epsilon$$

since $\mathcal{L}_0 \leq \mathcal{L}$, and where

$$\begin{aligned} A_0 = & \operatorname{Re}\{1050 \cdot P_4\left(\frac{\lambda}{\lambda + i\mu_1}\right) - 1365 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + i\mu_1}\right)\} \\ & + \operatorname{Re}\{745 \cdot P_4\left(\frac{\lambda}{\lambda + 2i\mu_1}\right) - 1400 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 2i\mu_1}\right)\} \\ & + \operatorname{Re}\{350 \cdot P_4\left(\frac{\lambda}{\lambda + 3i\mu_1}\right) - 870 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 3i\mu_1}\right)\} \\ & + \operatorname{Re}\{125 \cdot P_4\left(\frac{\lambda}{\lambda + 4i\mu_1}\right) - 350 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1 + 4i\mu_1}\right)\}. \end{aligned}$$

Applying Lemma 5.6 to each term of A_0 , it follows that $A_0 \leq 0$ provided

$$(10.10) \quad \frac{1050}{\lambda^4} - \frac{1365}{(\lambda + \lambda_1)^4} \leq 0.$$

Thus, (10.9) implies

$$0 \leq 620 \cdot P_4(1) - 1050 \cdot P_4\left(\frac{\lambda}{\lambda + \lambda_1}\right) + 2890\phi\lambda + \epsilon$$

provided (10.10) is satisfied. Taking $\lambda = 1.291$ yields $\lambda_1 \geq 0.0875$. This completes the proof of Theorem 1.1. \square

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